Complex Analysis

from the context of the course
MTH 425: Complex Analysis

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Contents

Chapter 1

Fundamentals

1.1 Complex Numbers

Definition 1.1.1. The set of **complex numbers** $\mathbb{C}$ is defined where $i^2 = -1$ by

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Definition 1.1.2. The **complex conjugate** of a complex number $a + bi = z \in \mathbb{C}$ denoted $\overline{z}$ is defined as $\overline{z} = a - bi$.

Definition 1.1.3. The **norm** of a complex number $z \in \mathbb{C}$ denoted $|z|$ is defined as $|z| = \sqrt{z\overline{z}}$.

Theorem 1.1.1. Euler's Formula states that for any real number $\varphi \in \mathbb{R}$,

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Proposition 1.1.1. Any complex number $z \in \mathbb{C}$ can be represented in the form $z = re^{i\phi}$.

Definition 1.1.4. The **nth roots of unity** are the complex numbers $e^{i2\pi k/n}$ for $k = 0, 1, \ldots, n-1$.

Definition 1.1.5. A **field** R is a set with two laws of composition denoted $+$ and $\times$ that satisfy the following axioms:

- **Identity** $\exists$ elements denoted $0, 1 \in R$ such that $1 \times a = a$ and $0 + a = a$, $\forall\ a \in R$.
- **Additive Inverse** For all $a \in R$, there exists an element $-a \in R$ such that $-a + a = 0$.
- **Multiplicative Inverse** For all nonzero $a \in F$, there exists an element $a^{-1} \in R$ such that $a \times a^{-1} = 1$.
- **Associativity** For all $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$ and $a + (b + c) = (a + b) + c$.
- **Commutativity** For all $a, b \in R$, $a \times b = b \times a$ and $a + b = b + a$.
- **Distributivity** For all $a, b, c \in R$, $a \times (b + c) = (a \times b) + (a \times c)$.

Proposition 1.1.2. The complex numbers $\mathbb{C}$ is a field with multiplicative inverses $z^{-1} = \frac{\overline{z}}{|z|^2}$ for any $z \in \mathbb{C}$.

Proposition 1.1.3. $\mathbb{R}$ is a subfield of $\mathbb{C}$.

Definition 1.1.6. A **domain** is an open and connected subset of $\mathbb{C}$.

Definition 1.1.7. An element $z \in \mathbb{C}$ is a **limit point** of a subset $S \subset \mathbb{C}$ if every neighborhood of z intersects $S - \{z\}$.

Definition 1.1.8. A subset $S \subset \mathbb{C}$ is

- **open** if for every $s \in S$ there is a neighborhood of s that is a subset of S.
- **closed** if every limit point of S is in S.
- **bounded** if for some real number $M \in \mathbb{R}$ and a point $z \in \mathbb{C}$, $S \subseteq B(x, M)$.
- **dense** in $X \subset \mathbb{C}$ if any $x \in X$ is a limit point of S.

1.2 Functions

Definition 1.2.1. A **function** $f : A \to B$ is a subset of $X \times Y$ such that $\forall x \in X$, $\exists$ exactly one element $y \in B$, $(x, y) \in f$.

Definition 1.2.2. The **domain** of a function $f : A \to B$ is $\{a \in A : \exists b \in B \text{ such that } (a, b) \in f\}$.

Definition 1.2.3. The **range** of a function $f : A \to B$ is $\{b \in B : \exists a \in A \text{ such that } (a, b) \in f\}$.

Definition 1.2.4. A function is a **injective** denoted $f : A \hookrightarrow B$ iff $f(x) = f(u) \Rightarrow x = y$.

Definition 1.2.5. A function is a **surjection** denoted $f : A \twoheadrightarrow B$ iff the range of f equals B.

Definition 1.2.6. A function is a **bijection** denoted $f : A \hookrightarrow\!\!\!\!\rightarrow B$ iff it is both an injection and a surjection.

Definition 1.2.7. The **limit** of a function f as $z \to z_0$ denoted $\lim_{z\to z_0} f(z) = \omega_0 \in \mathbb{C}$ iff $\forall\ \varepsilon > 0\ \exists\delta > 0$ such that $0 \leq |x - x_0| < \delta \ \Rightarrow\ |f(z) - \omega_0| < \varepsilon$.

Definition 1.2.8. A function is **continuous at** $z_0 \in \mathbb{C}$ iff $\lim_{z\to z_0} f(z) = f(z_0)$.

Definition 1.2.9. A function is **continuous** iff it is continuous at all points.

1.3 Differentiation

Definition 1.3.1. The **complex derivative** of a function f denoted $\frac{\partial}{\partial z} f(z)$ is defined

$$\frac{\partial}{\partial z} f(z) = \lim_{h\to 0} \frac{f(z + h) - f(z)}{h}$$

Definition 1.3.2. A function is **differentiable** iff the derivative exists for all $z \in \mathbb{C}$.

Proposition 1.3.1. If $f(x + iy) = u(x + iy) + iv(x + iy)$ is differentiable if and only if the partial derivatives exist and the **Cauchy-Riemann equations** hold

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 1.3.3. A function f is **analytic** on an open set $S \subset \mathbb{C}$ iff f is differentiable at every point $s \in S$.

Definition 1.3.4. A set S is **connected** iff any two points $a, b \in S$ there exists a continuous function $p : [0, 1] \to S$ such that $p(0) = a$ and $p(1) = b$.

Theorem 1.3.1. If f is analytic on $S \subset \mathbb{C}$ and $\frac{\partial}{\partial z} f(z) = 0$ for all $z \in S$, then for some constant $a \in \mathbb{C}$, $f(z) = a$ for all $z \in S$.

1.4 Harmonic Functions

Definition 1.4.1. A function $f : D \to \mathbb{R}^2$ is **harmonic** where $D \subset \mathbb{R}^2$ iff f is C^2 continuous and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Theorem 1.4.1. A function $f = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic on $D \subset \mathbb{C}$ and u, v are C^2 continuous on D if and only if u and v are harmonic on D.

Definition 1.4.2. Two functions $u, v : \mathbb{C} \to \mathbb{R}$ are **harmonic conjugates** on $D \subset \mathbb{C}$ iff $f = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic on D and u, v are C^2 continuous on D.

1.5 Polynomials

Definition 1.5.1. A **degree n polynomial** is a function $p : \mathbb{C} \to \mathbb{C}$ defined by

$$p(z) = \sum_{m=0}^{n} a_m z^m$$

Definition 1.5.2. A êxtbfrational function is a function $p/q : \mathbb{C} \to \mathbb{C}$ where p and q are polynomials defined by

$$f(z) = \frac{p(z)}{q(z)}$$

Definition 1.5.3. A function is **λ-periodic** on a subset of $S \subseteq \mathbb{C}$ iff

$$f(z) = f(z + \lambda) \quad \forall z \in S$$

Proposition 1.5.1. e^z is $2\pi i$ periodic.

1.6 Standard Functions

Definition 1.6.1. The **sin** and **cos** functions $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are defined

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Proposition 1.6.1. The sin and cos functions are 2π-periodic.

Definition 1.6.2. The **argument function** $\arg_\tau : \mathbb{C} \to (\tau, \tau + 2\pi]$ is defined for $z \in \mathbb{C}$ by $z = |z|e^{i\arg_\tau(z)}$

Definition 1.6.3. The **standard argument function** $\mathrm{Arg} : \mathbb{C} \to (-\pi, \pi]$ is defined as $\mathrm{Arg}(z) = \arg_{-\pi}(z)$.

Definition 1.6.4. The **logarithm function** $\log_\tau : \mathbb{C} - \{0\} \to \mathbb{C}$ is defined as $\log(z) = \ln|z| + i\arg_\tau(z)$.

Definition 1.6.5. The **standard logarithm function** $\mathrm{Log} : \mathbb{C} - \{0\} \to \mathbb{C}$ is defined as $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$.

Definition 1.6.6. The **exponential function** $\wedge : \mathbb{C}^2 \to \mathbb{C}$ is defined for $(\alpha, \beta) \in \mathbb{C}^2$ as

$$\alpha^\beta = e^{\beta \log(\alpha)} = e^{\beta(\log|\alpha| + i \arg(\alpha))} = e^{\beta(\log|\alpha| + i\arg(\alpha) + i2\pi k)}$$

Definition 1.6.7. The **standard exponential function** $\wedge : \mathbb{C}^2 \to \mathbb{C}$ is defined for $(\alpha, \beta) \in \mathbb{C}^2$ as

$$\alpha^\beta = e^{\beta \log(\alpha)} = e^{\beta(\mathrm{Log}|\alpha| + i \arg(\alpha))} = e^{\beta(\mathrm{Log}|\alpha| + i\mathrm{Arg}(\alpha) + i2\pi k)}$$

Proposition 1.6.2. For any $\log_\tau : \mathbb{C} - \{0\} \to \mathbb{C}$ and any $z_1, z_2 \in \mathbb{C}$,

$$\log_\tau(z_1 z_2) = \log_\tau(z_1) + \log_\tau(z_2)$$

$$\log_\tau(z_1/z_2) = \log_\tau(z_1) - \log_\tau(z_2)$$

Theorem 1.6.1. The standard logarithm function is analytic on $\mathbb{C} - (-\infty, 0]$ and $\frac{\partial}{\partial z}\mathrm{Log}(z) = \frac{1}{z}$.

1.7 Integration

Definition 1.7.1. The **complex integral** denoted $\int_a^b f(t)dt$ of a continuous complex function $f : [a, b] \to \mathbb{C}$ such that $f(t) = u(t) + iv(t)$ is defined

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Theorem 1.7.1. The Fundamental Theorem of Calculus states that for any function $g : [a, b] \to \mathbb{C}$ if there exists $F : [a, b] \to \mathbb{C}$ such that $F'(t) = f(t)$, $\forall t \in [a, b]$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

Definition 1.7.2. A **smooth curve** is a function $f : [a, b] \to \mathbb{C}$, $f = u(t) + iv(t)$ iff

- f has a continuous derivative.
- f' is non-zero.

Definition 1.7.3. A **smooth closed curve** is a smooth curve $f : [a, b] \to \mathbb{C}$ such that

- $f(a) = f(b)$, $f'(a) = f'(b)$.
- f is bijective on $[a, b)$.

Definition 1.7.4. A **directed smooth curve** is a smooth curve $f : [a, b] \to \mathbb{C}$ where a is declared as the initial point.

Definition 1.7.5. A **directed smooth closed curve** is a smooth curve that is both directed and closed.

Definition 1.7.6. The **integral over a curve** γ with any parameterization $g_\gamma : [a, b] \to \mathbb{C}$ that is a directed smooth curve of a functoin $f : \mathbb{C} \to \mathbb{C}$ is defined

$$\int_\gamma f(z)dz = \int_a^b f(g(t))g'(t)dt$$

Definition 1.7.7. A **contour** is a finite collection of smooth curves connected at initial and final points.

Definition 1.7.8. A **loop contour** is a contour $\Gamma : [a, b] \to \mathbb{C}$ such that $\Gamma(a) = \Gamma(b)$.

Definition 1.7.9. A **simple contour** is a contour $\Gamma : [a, b] \to \mathbb{C}$ where there does not exist an element $(t_1, t_2) \in [a, b] \times (a, b)$ such that $\Gamma(t_1) = \Gamma(t_2)$.

Definition 1.7.10. The **integral over a contour** γ with component curves $\{\gamma_1, \ldots, \gamma_n\}$ of a function $f : \mathbb{C} \to \mathbb{C}$ is defined

$$\int_\Gamma f(z)dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)dz$$

Theorem 1.7.2. Let D be a domain. If continuous function $f : D \to \mathbb{C}$ has an anti derivative F, then for any $z_I, z_F \in D$ and any contour $\Gamma \subset D$ with initial point z_I and final point z_F the integral

$$\int_\Gamma f(z)dz = F(z_F) - F(z_I)$$

Corollary 1.7.2.1. Integrals of closed contours on continuous functions with anti derivatives are zero.

Theorem 1.7.3. Let f be continuous in a domain D, the following are equivalent

- f has an anti-derivative.
- Integrals of closed contours are zero.
- Contours that share initial and final points are equivalent.

1.8 Interior and Exterior

Definition 1.8.1. The **interior** of a simple closed contour Γ is the bounded subset $\text{Int}(\Gamma) \subset \mathbb{C}$ separated from $\mathbb{C} - \text{Int}(\Gamma)$ by the contour Γ.

Definition 1.8.2. The **exterior** of a simple closed contour Γ is the unbounded subset $\text{Ext}(\Gamma) \subset \mathbb{C}$ separated from $\mathbb{C} - \text{Ext}(\Gamma)$ by the contour Γ.

Theorem 1.8.1. The Jordan Curve Theorem states that any simple closed contour Γ separates $\mathbb{C}$ into an interior and exterior.

Definition 1.8.3. A contour Γ is **positively oriented** iff the interior is to the left of a point traveling along Γ.

Definition 1.8.4. A contour Γ is **negatively oriented** iff the interior is to the right of a point traveling along Γ.

Definition 1.8.5. A domain D is **simply connected** iff $\forall \Gamma \subset D$ if Γ is a simple closed contour then $\text{Int}(\Gamma) \subset D$.

Theorem 1.8.2. Let D be a simply connected domain, Γ be any simple loop contour, and f be any analytic function on D, then

$$\oint_\Gamma f(z)dz = 0$$

Definition 1.8.6. A contour $\Gamma_0 \subset D$ can be **continuously deformed** to another contour $\Gamma_1 \subset D$ iff there exists a continuous function $z : [0, 1] \times [0, 1] \to D$ such that

$$z(0, t) = \Gamma_0(t), \text{ and} z(1, t) = \Gamma_1(t)$$

Theorem 1.8.3. For D domain, let $\Gamma_0, \Gamma_1 \subset D$ be closed contours such that Γ_0 can be continuously transformed onto Γ_1 and f be an analytic function in D then

$$\oint_{\Gamma_0} f(z)dz = \oint_{\Gamma_1} f(z)dz$$

1.9 Cauchy's Integral Formula

Theorem 1.9.1. Cauchy's Integral Formula states that for any analytic function f on a simply connected domain D, if $\Gamma \subset D$ is a simple closed positively oriented contour, then for any $z_0 \subset \mathrm{Int}(\Gamma)$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Theorem 1.9.2. Morera's Theorem states that if f is continuous in a domain D such that

$$\int_\Gamma f(z)dz = 0, \quad \forall \Gamma \text{ loops in } D$$

then f is analytic in D.

Proposition 1.9.1. Cauchy estimates states for f analytic on $B_R(z_0)$,

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \max_{z \in B_R(z_0)} |f(z)|$$

Theorem 1.9.3. If f is bounded and holomorphic on all of $\mathbb{C}$, then f is constant.

Theorem 1.9.4. The Fundamental Theorem of Algebra states that every non-constant polynomial with complex coefficients has at least one zero.

Chapter 2

Complex Series

2.1 Power Series

Theorem 2.1.1. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

there exists $R \geq 0$ such that

1. If $R > 0$ then the series converges to an analytic function for $|z - z_0| < R$.
2. The series diverges for $|z - z_0| > R$.
3. $f'(z) = \sum_{n=1}^{\infty} a_n n (z - z_0)^{n-1}$ for $|z - z_0| < R$
4. If $\Gamma \subset B_R(z_0)$ is a contour then

$$\int_{\Gamma} f(z) dz = \sum_{n=0}^{\infty} a_n \int_{\Gamma} (z - z_0)^n dz$$

Definition 2.1.1. The **radius of convergence** of a power series is the real number $R \geq 0$ such that the properties in the previous theorem hold.

Theorem 2.1.2. Taylors Theorem states that for any analytic function f on domain D and $z_0 \in D$ for

$$a_n = \frac{f^{(n)(z_0)}}{n!} = \frac{1}{n\pi i} \int_{C_{R/2}(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

the series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to $f(z)$ for any $z \in B_R(z_0) \subset D$.

Theorem 2.1.3. Let f be analytic on $A = \{z | 0 < r < |z - z_0|\}$ then $\forall z \in A$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

for $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(\omega - z_0)^{n+1} d\omega}$ for $n = 0, 1, 2, \ldots$ and where $\Gamma \subset A$ is any closed simple positively oriented contour with $z_0 \in \mathrm{Int}(\Gamma)$.

Theorem 2.1.4. A series of the form

$$\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

that converges on A defines an analytic function on A.