Complex Analysis

from the context of the course
MTH 425: Complex Analysis

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Contents

Chapter 1

Fundamentals

1.1 Complex Numbers

Definition 1.1.1. The set of **complex numbers** $\mathbb{C}$ is defined where $i^2 = -1$ by

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Definition 1.1.2. The **complex conjugate** of a complex number $a + bi = z \in \mathbb{C}$ denoted $\overline{z}$ is defined as $\overline{z} = a - bi$.

Definition 1.1.3. The **norm** of a complex number $z \in \mathbb{C}$ denoted $|z|$ is defined as $|z| = \sqrt{z\overline{z}}$.

Theorem 1.1.1. Euler's Formula states that for any real number $\varphi \in \mathbb{R}$,

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Proposition 1.1.1. Any complex number $z \in \mathbb{C}$ can be represented in the form $z = re^{i\phi}$.

Definition 1.1.4. The **nth roots of unity** are the complex numbers $e^{i2/n}$ for $k = 0, 1, \ldots, n-1$.

Definition 1.1.5. A **field** R is a set with two laws of composition denoted $+$ and $\times$ that satisfy the following axioms:

- **Identity** $\exists$ elements denoted $0, 1 \in R$ such that $1 \times a = a$ and $0 + a = a$, $\forall\ a \in R$.
- **Additive Inverse** For all $a \in R$, there exists an element $-a \in R$ such that $-a + a = 0$.
- **Multiplicative Inverse** For all nonzero $a \in F$, there exists an element $a^{-1} \in R$ such that $a \times a^{-1} = 1$.
- **Associativity** For all $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$ and $a + (b + c) = (a + b) + c$.
- **Commutativity** For all $a, b \in R$, $a \times b = b \times a$ and $a + b = b + a$.
- **Distributivity** For all $a, b, c \in R$, $a \times (b + c) = (a \times b) + (a \times c)$.

Proposition 1.1.2. The complex numbers $\mathbb{C}$ is a field with multiplicative inverses $z^{-1} = \frac{\overline{z}}{|z|^2}$ for any $z \in \mathbb{C}$.

Proposition 1.1.3. $\mathbb{R}$ is a subfield of $\mathbb{C}$.

1.2 Functions

Definition 1.2.1. A **function** $f : A \to B$ is a subset of $X \times Y$ such that $\forall x \in X$, $\exists$ exactly one element $y \in B$, $(x, y) \in f$.

Definition 1.2.2. The **domain** of a function $f : A \to B$ is $\{a \in A : \exists b \in B \text{ such that } (a, b) \in f\}$.

Definition 1.2.3. The **range** of a function $f : A \to B$ is $\{b \in B : \exists a \in A \text{ such that } (a, b) \in f\}$.

Definition 1.2.4. A function is a **injective** denoted $f : A \hookrightarrow B$ iff $f(x) = f(u) \Rightarrow x = y$.

Definition 1.2.5. A function is a **surjection** denoted $f : A \twoheadrightarrow B$ iff the range of f equals B.

Definition 1.2.6. A function is a **bijection** denoted $f : A \hookrightarrow\!\!\!\!\rightarrow B$ iff it is both an injection and a surjection.

"Then he became a philosopher, which is very sad."