Fourier Analysis

from the context of the course
MTH 496: Capstone in Fourier Analysis

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Contents

Chapter 1

Introduction

1.1 Complex Numbers

Definition 1.1.1. The set of **complex numbers** $\mathbb{C}$ is defined where $i^2 = -1$ by

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Definition 1.1.2. The **complex conjugate** of a complex number $a + bi = z \in \mathbb{C}$ denoted $\overline{z}$ is defined as $\overline{z} = a - bi$.

Definition 1.1.3. The **norm** of a complex number $z \in \mathbb{C}$ denoted $|z|$ is defined as $|z| = \sqrt{z\overline{z}}$.

Theorem 1.1.1. Taylor's Theorem states that any C^∞ continuous function $f : \mathbb{C} \to \mathbb{C}$ can be represented as a Taylor series centered at any point $a \in \mathbb{C}$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n$$

Theorem 1.1.2. Euler's Formula states that for any real number $\varphi \in \mathbb{R}$,

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Proposition 1.1.1. Any complex number $z \in \mathbb{C}$ can be represented in the form $z = re^{i\phi}$.

Definition 1.1.4. A **field** R is a set with two laws of composition denoted $+$ and $\times$ that satisfy the following axioms:

- **Identity** $\exists$ elements denoted $0, 1 \in R$ such that $1 \times a = a$ and $0 + a = a$, $\forall\ a \in R$.
- **Additive Inverse** For all $a \in R$, there exists an element $-a \in R$ such that $-a + a = 0$.
- **Multiplicative Inverse** For all nonzero $a \in F$, there exists an element $a^{-1} \in R$ such that $a \times a^{-1} = 1$.
- **Associativity** For all $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$ and $a + (b + c) = (a + b) + c$.
- **Commutativity** For all $a, b \in R$, $a \times b = b \times a$ and $a + b = b + a$.
- **Distributivity** For all $a, b, c \in R$, $a \times (b + c) = (a \times b) + (a \times c)$.

Proposition 1.1.2. The complex numbers $\mathbb{C}$ is a field with multiplicative inverses $z^{-1} = \frac{\overline{z}}{|z|^2}$ for any $z \in \mathbb{C}$.

Proposition 1.1.3. $\mathbb{R}$ is a subfield of $\mathbb{C}$.

1.2 Complex Vectors

Definition 1.2.1. The **index set** of size $N \in \mathbb{N}$ denoted $[N]$ is the set $[N] = \{0, \ldots, N - 1\}$.

Definition 1.2.2. the **Nth complex vector space** is the set $\mathbb{C}^N$ of vectors $\mathbf{v} \in \mathbb{C}^N$ with N complex components $v_j \in \mathbb{C}$.

Definition 1.2.3. The **dot product** or **Hadamard product** of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ denoted $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$(\mathbf{u} \cdot \mathbf{v})_j = u_j \cdot v_j$$

1.3 Discrete Fourier Transform

Definition 1.3.1. The **discrete Fourier transform (DFT)** is a linear transformation $F : \mathbb{C}^n \to \mathbb{C}^n$ defined with the $n \times n$ matrix

$$F_{\omega,j} = \frac{1}{\sqrt{n}} e^{-2\pi i \omega j/n}$$

The DFT of a vector $\mathbf{v} \in \mathbb{C}^n$ is $\hat{\mathbf{v}} = F\mathbf{v} \in \mathbb{C}^n$.

Definition 1.3.2. A linear transformation is unity iff $U^\dagger U = I$.

Proposition 1.3.1. The DFT matrix F is unitary.

Definition 1.3.3. The **inverse discrete Fourier transform (IDFT)** is the inverse $F^{-1} = F^\dagger$.

1.4 Convolutions

Definition 1.4.1. The **circular convolution** of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ denoted $\mathbf{u} * \mathbf{v}$ is defined as

$$(\mathbf{u} * \mathbf{v})_j = \sum_{j=0}^{N-1} u_j v_{(k-j) \mod N}$$

Corollary 1.4.0.1. The circular convolution is commutative.

Definition 1.4.2. A matrix is **circulant** iff all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector.

Proposition 1.4.1. The circular convolution by a vector $\mathbf{v} \in \mathbb{C}^N$ can be represented as a circulant matrix:

$$\mathrm{circ}(\mathbf{v})_{kj} = v_{(j-k) \mod N}$$

Theorem 1.4.1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ then $\left(\widehat{\mathbf{u} * \mathbf{v}}\right)_\omega = \left(F\left(\mathbf{u} * \mathbf{v}\right)\right)_\omega = \sqrt{N}\widehat{u}_\omega \cdot \widehat{v}_\omega$, $\forall \omega \in [N]$.

Corollary 1.4.1.1. For any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$, $\mathbf{u} * \mathbf{v} = \sqrt{N} F^\dagger (F\mathbf{u} \cdot F\mathbf{v}) = \sqrt{N}(\widehat{\widehat{\mathbf{u}} \cdot \widehat{\mathbf{v}}})$.

Corollary 1.4.1.2. For any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$,

$$\widehat{u}_\omega = \left(F\mathbf{u}\right)_\omega = \frac{\left(F\left(\mathbf{u} * \mathbf{v}\right)\right)_\omega}{(F\mathbf{b})_\omega \sqrt{N}} = \frac{\left(\widehat{\mathbf{u} * \mathbf{v}}\right)_\omega}{\widehat{b}_\omega \sqrt{N}}$$

Theorem 1.4.2. The eigenvectors of $circ(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{C}^N$ are the columns of $F^\dagger$.

Proposition 1.4.2. For two polynomials $g(x) = \sum_{j=0}^{N-1} g_j x^j$ and $r(x) = \sum_{j=0}^{N-1} r_j x^j$. The product of the two polynomials has at most degree $2N - 2$ and can be computed by convolving the two polynomial vectors padded with $N - 1$ zeros.

$$g(x)r(x) = \mathbf{g} * \mathbf{r} = \begin{pmatrix} g_0 \\ \vdots \\ g_{N-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} * \begin{pmatrix} r_0 \\ \vdots \\ r_{N-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Theorem 1.4.3. The **fast Fourier transform** is an algorithm for quickly computing the Fourier transform of a vector $\mathbf{u} \in \mathbb{C}^N$. Let $N = p_1 p_2 \ldots p_m$ be the unique prime factorization of N.

$$\widehat{u}_\omega = \frac{1}{\sqrt{N}} \sum_{k=0}^{p_1-1} \widehat{u^{(k,p_1)}}_{\omega \mod \frac{N}{p_1}} e^{-2\pi i \omega k/N}$$

"You're apparently majoring in mathematics, and this is the most important topic in mathematics, so I win!"
"When N is up to you, use a power of two!"