Fourier Analysis

from the context of the course
MTH 496: Capstone in Fourier Analysis

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Contents

Chapter 1

Discrete Fourier Analysis

1.1 Complex Numbers

Definition 1.1.1. The set of **complex numbers** $\mathbb{C}$ is defined where $i^2 = -1$ by

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Definition 1.1.2. The **complex conjugate** of a complex number $a + bi = z \in \mathbb{C}$ denoted $\overline{z}$ is defined as $\overline{z} = a - bi$.

Definition 1.1.3. The **norm** of a complex number $z \in \mathbb{C}$ denoted $|z|$ is defined as $|z| = \sqrt{z\overline{z}}$.

Theorem 1.1.1. Taylor's Theorem states that any C^∞ continuous function $f : \mathbb{C} \to \mathbb{C}$ can be represented as a Taylor series centered at any point $a \in \mathbb{C}$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n$$

Theorem 1.1.2. Euler's Formula states that for any real number $\varphi \in \mathbb{R}$,

$$e^{i\phi} = \cos\phi + i \sin\phi$$

Proposition 1.1.1. Any complex number $z \in \mathbb{C}$ can be represented in the form $z = re^{i\phi}$.

Definition 1.1.4. A **field** R is a set with two laws of composition denoted $+$ and $\times$ that satisfy the following axioms:

- **Identity** $\exists$ elements denoted $0, 1 \in R$ such that $1 \times a = a$ and $0 + a = a$, $\forall\ a \in R$.
- **Additive Inverse** For all $a \in R$, there exists an element $-a \in R$ such that $-a + a = 0$.
- **Multiplicative Inverse** For all nonzero $a \in F$, there exists an element $a^{-1} \in R$ such that $a \times a^{-1} = 1$.
- **Associativity** For all $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$ and $a + (b + c) = (a + b) + c$.
- **Commutativity** For all $a, b \in R$, $a \times b = b \times a$ and $a + b = b + a$.
- **Distributivity** For all $a, b, c \in R$, $a \times (b + c) = (a \times b) + (a \times c)$.

Proposition 1.1.2. The complex numbers $\mathbb{C}$ is a field with multiplicative inverses $z^{-1} = \frac{\overline{z}}{|z|^2}$ for any $z \in \mathbb{C}$.

Proposition 1.1.3. $\mathbb{R}$ is a subfield of $\mathbb{C}$.

1.2 Complex Vectors

Definition 1.2.1. The **index set** of size $N \in \mathbb{N}$ denoted $[N]$ is the set $[N] = \{0, \ldots, N - 1\}$.

Definition 1.2.2. the **Nth complex vector space** is the set $\mathbb{C}^N$ of vectors $\mathbf{v} \in \mathbb{C}^N$ with N complex components $v_j \in \mathbb{C}$.

Definition 1.2.3. The **dot product** or **Hadamard product** of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ denoted $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$(\mathbf{u} \cdot \mathbf{v})_j = u_j \cdot v_j$$

1.3 Discrete Fourier Transform

Definition 1.3.1. The **discrete Fourier transform (DFT)** is a linear transformation $F : \mathbb{C}^n \to \mathbb{C}^n$ defined with the $n \times n$ matrix

$$F_{\omega,j} = \frac{1}{\sqrt{n}} e^{-2\pi i \omega j/n}$$

The DFT of a vector $\mathbf{v} \in \mathbb{C}^n$ is $\hat{\mathbf{v}} = F\mathbf{v} \in \mathbb{C}^n$.

Definition 1.3.2. A linear transformation is unity iff $U^\dagger U = I$.

Proposition 1.3.1. The DFT matrix F is unitary.

Definition 1.3.3. The **inverse discrete Fourier transform (IDFT)** is the inverse $F^{-1} = F^\dagger$.

Theorem 1.3.1. The **fast Fourier transform** is an algorithm for quickly computing the Fourier transform of a vector $\mathbf{u} \in \mathbb{C}^N$. Let $N = p_1 p_2 \ldots p_m$ be the unique prime factorization of N.

$$\widehat{u}_\omega = \frac{1}{\sqrt{N}} \sum_{k=0}^{p_1-1} \widehat{u^{(k,p_1)}}_{\omega \mod \frac{N}{p_1}} e^{-2\pi i \omega k/N}$$

Result 1.3.1. The fomplexity of the fast Fourier transform for $N = 2^m$ is $\mathcal{O}(N \log N)$

1.4 Convolutions

Definition 1.4.1. The **circular convolution** of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ denoted $\mathbf{u} * \mathbf{v}$ is defined as

$$(\mathbf{u} * \mathbf{v})_k = \sum_{j=0}^{N-1} u_j v_{(k-j) \mod N}$$

Corollary 1.4.0.1. The circular convolution is commutative.

Definition 1.4.2. A matrix is **circulant** iff all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector.

Proposition 1.4.1. The circular convolution by a vector $\mathbf{v} \in \mathbb{C}^N$ can be represented as a circulant matrix:

$$\text{circ}(\mathbf{v})_{kj} = v_{(j-k) \mod N}$$

Theorem 1.4.1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ then $\left(\widehat{\mathbf{u} * \mathbf{v}}\right)_\omega = \left(F\left(\mathbf{u} * \mathbf{v}\right)\right)_\omega = \sqrt{N}\widehat{u}_\omega \cdot \widehat{v}_\omega$, $\forall \omega \in [N]$.

Corollary 1.4.1.1. For any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$, $\mathbf{u} * \mathbf{v} = \sqrt{N} F^\dagger (F\mathbf{u} \cdot F\mathbf{v}) = \sqrt{N}(\widehat{\widehat{\mathbf{u}} \cdot \widehat{\mathbf{v}}})$.

Corollary 1.4.1.2. For any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$,

$$\widehat{u}_\omega = (F\mathbf{u})_\omega = \frac{\left(F\left(\mathbf{u} * \mathbf{v}\right)\right)_\omega}{(F\mathbf{b})_\omega \sqrt{N}} = \frac{\left(\widehat{\mathbf{u} * \mathbf{v}}\right)_\omega}{\widehat{b}_\omega \sqrt{N}}$$

Theorem 1.4.2. The eigenvectors of $circ(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{C}^N$ are the columns of $F^\dagger$.

Proposition 1.4.2. For two polynomials $g(x) = \sum_{j=0}^{N-1} g_j x^j$ and $r(x) = \sum_{j=0}^{N-1} r_j x^j$. The product of the two polynomials has at most degree $2N - 2$ and can be computed by convolving the two polynomial vectors padded with $N - 1$ zeros.

$$g(x)r(x) = \mathbf{g} * \mathbf{r} = \begin{pmatrix} g_0 \\ \vdots \\ g_{N-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} * \begin{pmatrix} r_0 \\ \vdots \\ r_{N-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

1.5 Trigonometric Polynomials

Definition 1.5.1. A **trigonometric polynomial** of degree N is a function $p : \mathbb{R} \to \mathbb{C}$ defined by

$$p(x) = \sum_{\omega=-N}^{N} p_\omega e^{-\omega x}$$

Corollary 1.5.0.1. A trigonometric polynomial is 2π periodic.

Definition 1.5.2. The **inner product of trigonometric polynomials** $p, q \in \mathbb{C}^{2N+1}$ is defined

$$\langle p, q\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)\overline{q(x)}dx = \sum_{\omega=-N}^{N} p_\omega \overline{q}_\omega$$

Definition 1.5.3. The **norm of a trigonometric polynomial** $p \in \mathbb{C}^{2N+1}$ is defined by

$$||p||^2 = \langle p, p\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} |p(x)|^2 dx = \sum_{\omega=-N}^{N} |p_\omega|^2$$

Definition 1.5.4. The **Fourier transform of a trigonometric polynomial** $p \in \mathbb{C}^{2N+1}$ is defined as

$$F[p](\omega) = \widehat{p}_\omega = \langle p, e^{i\omega x}\rangle = p_\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)e^{-i\omega x}dx$$

Definition 1.5.5. The **Inverse Fourier transform of a trigonometric polynomial** $\widehat{p}_\omega \in \mathbb{C}^{2N+1}$ is defined as

$$F^{-1}[\widehat{p}](x) = \sum_{\omega=-N}^{N} \widehat{p}_\omega e^{i\omega x}$$

Definition 1.5.6. The **convolution of two trigonometric polynomials** $p, q \in \mathbb{C}^{2N+1}$ denoted $p * q$ is defined

$$(p * q)(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)q(y - x)dx$$

Theorem 1.5.1. $F[p * q](\omega) = F[p](\omega) \cdot F[q](\omega) = \widehat{p}_\omega \cdot \widehat{q}_\omega \quad \forall\omega\{-N, \ldots, N\}$

Corollary 1.5.1.1. $\deg(p * q) \leq \min\{\deg(p), \deg(q)\}$

Theorem 1.5.2. Let $p \in \mathbb{C}^{2N+1}$ be a trigonometric polynomial with Fourier transform $\widehat{p}_\omega$. The discrete Fourier transform $\widehat{v}$ of the vector v defined by $v_j = p\left(\frac{2\pi j}{2N+1}\right)$ is the reordered elements of $\widehat{p}_\omega$.

$$\begin{pmatrix} \widehat{p}_0 \\ \vdots \\ \widehat{p}_N \\ \widehat{p}_{-N} \\ \widehat{p}_{-N+1} \\ \vdots \\ \widehat{p}_{-1} \end{pmatrix} = \frac{\widehat{v}}{\sqrt{2N+1}}$$

Corollary 1.5.2.1. For a trigonometric polynomial $p \in \mathbb{C}^{2N+1}$, $\widehat{p}_\omega = \frac{2\pi}{2N+1}\sum_{j=0}^{2N} p\left(\frac{2\pi j}{2N+1}\right) e^{\frac{-2\pi i j\omega}{2N+1}}$

Theorem 1.5.3. For an infinite trigonometric polynomial $p \in \mathbb{C}^\infty$, defined by $p(x) = \sum_{\omega\in\mathbb{Z}} c_\omega e^{i\omega x}$ and a finite sampling $u \in \mathbb{C}^N$ defined by $c_j = p\left(\frac{2\pi j}{N}\right)$, the Fourier transform of u is

$$\widehat{u}_\omega = \sqrt{N} \sum_{j=\omega \mod N} \widehat{p}_j$$

1.6 Heat Equation

Definition 1.6.1. The **heat equation** is a differential equation of the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2}{\partial x^2}u - f$$

where $u(x,t), u(x,0)$, and $f(x)$ are degree $\leq N$ trigonometric polynomials.

$$u(x,t) = \sum_{\omega=-N}^{N} \widehat{u}_\omega(t)e^{i\omega x}$$

Proposition 1.6.1. The derivative of the Fourier transform can be written as

$$\frac{\partial}{\partial t}\widehat{u}_\omega(t) = -k\omega^2\widehat{u}_\omega(t) - \widehat{f}_\omega$$

Proposition 1.6.2. For $\omega \neq 0$ the Fourier transform of the solution to the heat equation is

$$\widehat{u}_\omega(t) = e^{-k\omega^2 t}\left(\widehat{u}_\omega(0) + \frac{\widehat{f}_\omega}{k\omega^2}\right) - \frac{\widehat{f}_\omega}{k\omega^2}$$

Theorem 1.6.1. The solution to the heat equation can be written in terms of discrete Fourier transforms

$$u(x,t) = \sum_{\omega=-N}^{N}\left(e^{-k\omega^2 t}\left(\overline{u}_\omega(0) + \frac{\widehat{f}_\omega}{k\omega^2}\right) - \frac{\widehat{f}_\omega}{k\omega^2}\right)e^{i\omega x} + \widehat{g}_o$$

Chapter 2

Fourier Series

2.1 Riemann Integrability

Definition 2.1.1. A **partition** P_N of $[a,b]$ is a set of N ordered points $P_N \subset [a,b]$ denoted $P_N = \{x_0, x_1, \dots, x_{N-1}\}$ such that

$$a = x_0 < x_1 < \dots < x_{N-1} = b$$

Definition 2.1.2. The **mesh size** of a partition P_N is $\max_{i\in[0,N-2]\cap\mathbb{Z}} |x_{i+1} - x_i|$

Definition 2.1.3. The **upper and lower sums** denoted $L(f, P_N)$ and $U(f, P_N)$ of a function f with partition P_N are defined

$$U(f, P_N) = \sum_{j=0}^{N-2} \sup_{t\in[x_j,x_{j+1}]} (x_{j+1} - x_j)$$

$$L(f, P_N) = \sum_{j=0}^{N-2} \inf_{t\in[x_j,x_{j+1}]} (x_{j+1} - x_j)$$

Definition 2.1.4. A function f is a **Riemann integrable function** iff $\forall\varepsilon > 0$, there exists P_N with $\varepsilon(P_N) < \varepsilon$ such that

$$|L(f, P_N) - U(f, P_N)| < \varepsilon$$

Definition 2.1.5. A complex function $f : [a,b] \to \mathbb{C}$ is a **complex Riemann integrable function** iff both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Riemann integrable.

Corollary 2.1.0.1. A continuous function is Riemann integrable.

Definition 2.1.6. A function $f : [a,b] \to \mathbb{C}$ is **piecewise continuous** denoted $f \in \mathrm{P}([a,b])$ iff there exists a finite set of points $\{x_0, x_1, \dots, x_{N-1}\} = D \subset [a,b]$ such that f is continuous on the following intervals

$$(a, x_0), (x_0, x_1), \dots, (x_{N-2}, x_{N-1}), (x_{N-1}, b)$$

Theorem 2.1.1. If a function f is piecewise continuous and bounded then it is Riemann integrable.

Definition 2.1.7. A function $f : [a,b] \to \mathbb{C}$ is a **piecewise smooth** denoted $f \in \mathrm{PS}([a,b])$ iff f, f' both exist and are piecewise continuous.

Definition 2.1.8. The **inner product of functions** $f, g : [a,b] \to \mathbb{C}$ is defined

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

2.2 Bessel's Inequality

Definition 2.2.1. The **Fourier coefficients** denoted c_n, a_n, b_n of a function $f : [a,b] \to \mathbb{C}$ are

$$c_n = \frac{1}{b-a}\int_a^b f(\theta)e^{\frac{-2\pi i n\theta}{b-a}}d\theta$$

$$a_n = \frac{2}{b-a}\int_a^b f(\theta)\cos\left(\frac{2\pi n\theta}{b-a}\right)d\theta$$

$$b_n = \frac{2}{b-a}\int_a^b f(\theta)\sin\left(\frac{2\pi n\theta}{b-a}\right)d\theta$$

Corollary 2.2.0.1. The Fourier coefficients for a 2π-periodic function $f : [-\pi, \pi] \to \mathbb{C}$ are

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta$$

Theorem 2.2.1. **Bessel's Inequality** states that if $f : [a, b] \to \mathbb{C}$ is Riemann integrable then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{b-a} \int_{-a}^{b} |f(\theta)|^2$$

Corollary 2.2.1.1. If $f : [-\pi, \pi] \to \mathbb{C}$ is 2π-periodic and Riemann integrable then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2$$

Corollary 2.2.1.2. The Fourier series converges absolutely.

Definition 2.2.2. The **Nth parital Fourier sum** of a function $f : [a, b] \to \mathbb{C}$ denoted $S_N^f(\theta)$ defined

$$S_N^f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$$

Theorem 2.2.2. If $f : [a, b] \to \mathbb{C}$ is a piecewise smooth function that is $f \in \mathrm{PS}([a, b])$ with discontinuities $\{x_0, x_1, \dots\}$ then

$$\lim_{N\to\infty} S_N^f(\theta) = f(\theta), \quad \forall \theta \in [a, b] - \{x_0, x_1, \dots\}$$

2.3 The Dirchlet Kernel

Definition 2.3.1. The **Dirchlet Kernel** denoted $D_N(y)$ is defined

$$D_N(y) = \frac{1}{2\pi} \sum_{k=-N}^{N} e^{iky}$$

Corollary 2.3.0.1. For any function $f : [a, b] \to \mathbb{C}$, $S_N^f(\theta) = (D_N * f)(\theta)$.

Proposition 2.3.1. The Dirchlet kernel can be written in terms of sin, $D_N(y) = \frac{\sin((N+1/2)y)}{\sin(y/2)}$

Proposition 2.3.2. The following properties of the Dirchlet kernel $D_N(y)$ hold for all $N \in \mathbb{N}$.

- $D_N(0) = 2N + 1$
- The Dirchlet kernel is 2π-periodic
- $|D_N(y)| \leq \frac{1}{|\sin(y/2)|}$
- $D_N\left(\frac{2\pi}{2N+1}k\right) = 0$ for $k \in \{-N, \dots, -1, 1, \dots, N\}$
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y) = 1$
- $\frac{1}{2\pi}\int_{-\pi}^{0} D_N(y) = \frac{1}{2\pi}\int_{0}^{\pi} D_N(y) = \frac{1}{2}$

Theorem 2.3.1. The S_N^f is piecewise continuous and bounded.

Theorem 2.3.2. If $f : [a, b] \to \mathbb{C}$ is a piecewise smooth function that is $f \in \mathrm{PS}([a, b])$ with discontinuities $\{x_0, x_1, \dots\}$ then

$$\lim_{N\to\infty} S_N^f(\theta) = \frac{f(\theta^-) + f(\theta^+)}{2}, \quad \forall \theta \in [a, b]$$

Proposition 2.3.3. The Cardinality of $PS([-\pi, \pi])$ is $|\mathbb{R}|$.

2.4 Smoothness

Definition 2.4.1. A function f is $\mathbf{C^k}$ **smooth** iff $f^{(k)}$ exists and is Riemann integrable.

Corollary 2.4.0.1. A function f is C^k smooth iff $f', f'', \ldots, f^{(k-1)}$ are continuous.

Proposition 2.4.1. For a C^1 smooth function $f \in C^1$ with Fourier coefficients c_n and derivative f' with Fourier coefficients c_n'.

$$c_n' = inc_n$$

Theorem 2.4.1. For a C^k smooth function $f \in C^k$ with Fourier coefficients c_n and derivatives $f^{(\ell)}$ with Fourier coefficients $c_n^{(\ell)}$.

$$c_n^{(k)} = (in)^\ell c_n, \quad \forall n \in \{0, 1, \ldots, k\}$$

Proposition 2.4.2. For a real valued function $f : [a, b] \to \mathbb{R}$, $c_n = \frac{a_n}{2} - \frac{ib_n}{2}$.

Corollary 2.4.1.1. For a C^k smooth function $f \in C^k$, there exists a sequence $a_n \to 0$ as $|n| \to \infty$ such that

$$|c_n| \leq a_n |n|^{-k}, \quad \forall n \in \mathbb{Z} - \{0\}$$

Theorem 2.4.2. For a C^k smooth function $f : [a, b] \to \mathbb{C} \ \in C^k$, there exists a constant $c \in \mathbb{R}^+$ such that

$$|f(\theta) - S_N^f(\theta)| \leq \frac{c}{N}, \quad \forall \theta \in [a, b]$$

Theorem 2.4.3. Let $g, f \in C^1$ be 2π-periodic functions then

$$(f * g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(\theta - y) dy, \quad \widehat{(f * g)}_n = \widehat{f}_n \cdot \widehat{g}_n$$

2.5 Aliasing Error

Definition 2.5.1. The **aliased Fourier coefficients** denoted $\tilde{c}_n$ are defined

$$\tilde{c}_n := \frac{(-1)^n}{N} \sum_{k=0}^{N-1} f\left(-\pi + k \cdot \frac{2\pi}{N}\right) e^{\frac{-2\pi i n k}{N}}$$

Theorem 2.5.1. Aliased Fourier coefficients $\tilde{c}_n$ with N samples can be written in terms of the exact Fourier coefficients c_n,

$$\tilde{c}_n = c_n + \sum_{|g|>1} c_{n+gN}$$

Chapter 3

Hilbert Spaces

3.1 Inner Product Spaces

Definition 3.1.1. A **vector space** is a set V with addition and scalar multiplication such that the folowing properties hold

- Commutativity: $v + w = w + v \ \forall \ v, w \in V$
- Associativity: $(u + v) + w = u + (w + v) \ \forall \ u, v, w \in V$
- Zero Vector: $\exists$ a vector 0 such that for any vector $v \in V, \ v + 0 = v$
- Additive Inverse: for any vector $v \in V$ there exists a vector $-v \in V$ such that $v - v = 0$
- Multiplicative Identity: for any vector $v \in V, v \cdot 1 = v$
- Additive Conservation: for any two vectors $v, w \in V, \ v + w \in V$
- Multiplicative Conservation: for any vector $v \in V$ and any scalar a, we have $av \in V$

Definition 3.1.2. An **inner product** denoted $\langle \cdot, \cdot, \rangle : V^2 \to \mathbb{C}$ is a function taking two vectors in a vector space V to $\mathbb{C}$ such that for all $u, v, w \in V$ and $\alpha \in \mathbb{C}$,

- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.
- $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$
- $\langle v, w\rangle = \widehat{\langle w, v\rangle}$
- $\langle v, v \in \mathbb{R}^+$ and $\langle v, v\rangle = 0 \ \Leftrightarrow \ v = 0$.

Definition 3.1.3. An **inner product space** is a vector space equipped with an inner product.

Definition 3.1.4. The **norm** of a vector $v \in V$ denoted $||v||$ is defined

$$||v|| = \sqrt{\langle v, v\rangle}$$

Proposition 3.1.1. For an inner product space V, any vectors $v, w \in V$, and any scalar $\alpha \in \mathbb{C}$,

- $||\alpha v|| = |\alpha|||v||$
- $||v|| = 0 \ \Leftrightarrow \ v = 0$
- $||v + w|| \leq ||v|| + ||w||$

Proposition 3.1.2. The **Cauchy Schwartz inequality** state that for any two vectors $v, w \in V$ in an inner product space,

$$|\langle v, w\rangle| \leq ||v||||w||$$

3.2 Hilbert Spaces

Definition 3.2.1. An **orthogonal** set of vectors $\{v_j\}_{j\in I} \subset V$ is a set of vectors in an inner product space such that

$$\langle v_j, v_\ell \rangle = \delta_{j,\ell}, \quad \forall j, \ell \in I$$

Theorem 3.2.1. General Bessel's Inequality states that for any vector $f \in V$ and any orthonormal set of vectors $\{v_j\}_{j\in I}$ in an inner product space,

$$\sum_{j\in I} |\langle f, v_j|^2 \leq ||f||^2$$

Definition 3.2.2. A **Cauchy sequence** of vectors $\{v_j\}_{j=1}^{\infty} \subset V$ is a sequence in an inner product space such that $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall n, m \geq N$, $||v_n - v_m|| \leq \varepsilon$.

Definition 3.2.3. A **complete inner product space** is an inner product space where every Cauchy sequence has a limit point.

Definition 3.2.4. A **Hilbert space** is a complete inner product space.

Definition 3.2.5. A sequence v_j **converges** in a Hilbert space H iff there exists $v \in H$ such that $\lim_{j\to\infty} ||v_j - v|| = 0$.

Definition 3.2.6. The **limit** of a convergent sequence v_j is defined as the vector $v \in H$ such that $\lim_{j\to\infty} ||v_j - v|| = 0$.

Proposition 3.2.1. The limit of a sequence is unique.

Definition 3.2.7. An orthonormal set of vectors $\{v_j\}_{j=1}^{N}$ is a **basis** iff

$$\sum_{j=1}^{\infty} \langle f, v_j \rangle v_j = f, \quad \forall f \in H$$

Proposition 3.2.2. For an orthonormal set of vectors $\{v_j\}_{j=1}^{N}$ and any vector $f \in H$,

$$\langle \sum_{j=1}^{\infty} \langle f, v_j \rangle v_j, v_\ell \rangle = \langle f, v_\ell \rangle$$

Theorem 3.2.2. Let $\{v_j\}_{j=1}^{N}$ be an orthonormal set of vectors in a Hilbert space H, the following are equivalent

- $\{v_j\}_{j=1}^{N}$ is an orthonormal basis of H.
- $\forall f \in H$, $f = \sum_{j=1}^{\infty} \langle f, v_j \rangle v_j$.
- $\langle f, v_j \rangle = 0$ for all j, if and only if $f = 0$.
- $\forall f \in H$, $||f||^2 = \sum_{j=1}^{\infty} |\langle v_j, f \rangle|^2$.

3.3 Sets of Functions

Definition 3.3.1. The **set of C^k smooth functions** denoted $C^k(D)$ is the set of continuous functions $f : D \to \mathbb{C}$ such that f has k continuous derivatives.

Definition 3.3.2. The **set of C^∞ smooth functions** denoted $C^\infty(D)$ is the set of continuous functions $f : D \to \mathbb{C}$ with infinitely many continuous derivatives.

Definition 3.3.3. The **set of Lebesgue integrable functions** denoted $L^p(D)$ is the set of functions $f : D \to \mathbb{C}$ such that

$$\int_D |f(x)|^p dx < \infty$$

Theorem 3.3.1. $L^2([a, b]) = \{f : [a, b] \to \mathbb{C} : \int_a^b |f(x)|^2 dx < \infty\}$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)} dx$$

Theorem 3.3.2. $\forall f \in L^2([a, b])$ and $\varepsilon > 0$ there exists $\tilde{f} \in C^\infty([a, b])$ such that $||f - \tilde{f}|| < \varepsilon$.

Theorem 3.3.3. $\{e^{inx}\}_{n\in\mathbb{Z}}$ is a basis in $L^2([a, b])$.

3.4 Convergence

Definition 3.4.1. A sequence of functions $\{f_n\}$ is **pointwise convergent** to a function $f : D \to \mathbb{C}$ iff for all $x \in D$,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

Definition 3.4.2. A sequence of functions $\{f_n\}$ is **uniform convergent** to a function $f : D \to \mathbb{C}$ iff for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in D$ and $n > N$,

$$|f_n(x) - f(x)| < \epsilon$$

3.5 Continuous Convolution and Fourier Transform

Definition 3.5.1. The **continuous convolution** $(f * g)$ of two functions f and g is defined

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

Theorem 3.5.1. For any $f, g \in L^1 \cap L^2$, $f * g \in L^1 \cap L^2$.

Definition 3.5.2. The **continuous Fourier transform** $\widehat{f}$ of a function f is defined

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Definition 3.5.3. $C_0 \subset L^2(\mathbb{R})$ is the subset of continuous in $L^2(\mathbb{R})$.

Theorem 3.5.2. Let $f \in L^2(\mathbb{R})$ then $\forall \varepsilon > 0$, there exists $\tilde{N} \in \mathbb{Z}^+$ and continuous $\tilde{f} \in L^2([-\tilde{N}, \tilde{N}]) \subset L^2(\mathbb{R})$ such that $||f - \tilde{f}|| \leq \varepsilon$.

Corollary 3.5.2.1. C_0 is dense in $L^2(\mathbb{R})$.

Theorem 3.5.3. Let $f, g \in L^1 \cap L^2$ and suppose that g has k bounded continuous derivatives $g', g'', \ldots, g^{(k)} \in L^1 \cap L^2$ then $f * g$ also has k-derivatives and

$$(f * g)^{(\ell)} = (f * g^{(k)})(x), \quad \forall \ell \in 0, 1, \ldots, k$$

"You're apparently majoring in mathematics, and this is the most important topic in mathematics, so I win!"
"When N is up to you, use a power of two!"
"And therefore we can conclude that everything is peachy."
"As long as its countable, like whatever who cares!"
"Let's anthropomorphize the Fourier series."
"You just have to be ready to go from zero to frickin' abstract!"
"Are you Hilbert's Friend? Then why are you in his space."
"I don't think any of you would do that, but you never know what your students are capable of."
"But how does this theorem make you feel?"