Net-based Real Analysis

from the context of the course

MTH 327H: Honors Intro Analysis

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Contents

Chapter 1

Notation

1.1 General

$\forall$ - For all
$\exists$ - Exists

1.1.1 Common Sets

$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
$\mathbb{N}$ - Set of all Natural Numbers

1.2 Set Notation

$\in$ - "In" := is an element of

Example. $\vec{v} \in \mathbb{R}^3$

$\notin$ - "Not In" := is not an element of

Example. $\vec{v} \notin \mathbb{R}^3$

$\{,\}$ - Set := elements of the set are listed inside the brackets
Example: $A = \{1, 2, 3\}$ "A is a set containing the elements 1, 2, and 3"
Note: elements in a set must be unique
$\{\}$ *or* $\emptyset$ - The Empty Set

Definition 1.2.1. $|\ |$ - **Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 1.2.2. $\cap$ - **Intersection** := The **Intersection** of two sets is the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}$

the **Intersection** of many sets can be denoted: $\cap_{i=1}^{k} A_i$*For the set of elements that appear in all of* $A_1 \cdots A_k$

Definition 1.2.3. $\cup$ - **Union** := The **Union** of two sets is the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}$

the **Union** of many sets can be denoted: $\cup_{i=1}^{k} A_i$ *For the set of elements that appear in any of* $A_1 \cdots A_k$

Definition 1.2.4. $\subseteq$ - **Subset** := Set A is a **Subset** of B if all the elements of A are also elements of B. We denote this by $A \subseteq B$.

Definition 1.2.5. $\subsetneq$ - **Proper Subset** := Set A is a **Proper Subset** of B if $A \subseteq B$ and $A \neq B$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

Chapter 2

Review of Set Theory

Definition 2.0.1. Two sets are consider to be equal if $A \subseteq B$ and $A \supseteq B$

Definition 2.0.2. Pairwise Disjoint := A set of sets $\Im$ is considered to be **Pairwise Disjoint** *if for* $S, T \in \Im$

$$S \neq T \Rightarrow S \cup T = \emptyset$$

There are two way of taking "differences" of sets:

$$X \backslash Y = \{x \in X : x \notin Y\}$$

$$X \Delta Y = (X \cup Y) \backslash (X \cap Y)$$

Definition 2.0.3. Given a set X and a set $\mathscr{S}$ whose elements are sets.

1. We say that $\mathscr{S}$ **covers** X if $X \subseteq \bigcup \mathscr{S}$
2. We say that $\mathscr{S}$ **partitions** X if $X = \bigcup \mathscr{S}$, the elements of $\mathscr{S}$ are non-empty, and $\mathscr{S}$ is pairwise disjoint

Definition 2.0.4. Ordered Pair (tuple) := an ordered list of two elements, each of which can be an arbitrary mathematical object and may or may not be the same. Denoted for $n \in \mathbb{N}$, an n-tuple is an ordered list of n elements, written as $(x_1, \ldots, x_n)$

Definition 2.0.5. For two sets X, Y the **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. More generally we can write a **Cartesian product** for n sets denoted by

$$X_1 \times X_2 \times \ldots \times X_n \ or \ \prod_{i=1}^{n} X_i$$

Remark. When taking the **Cartesian product** of the same set we use the shorthand: X^n

Remark. Additionally, the notation 2^X indicates the set of all possible subsets of X

Definition 2.0.6. We say that the **diagonal** of X^n is the subset $\{(x_1, \ldots, x_n) \in X^n : x_1 = x_2 = \ldots = x_n\}$

Definition 2.0.7. Given two sets X, Y we say that f is a **function** with domain X and codomain Y denoted $f : X \to Y$, if f is a subset of $X \times Y$ such that every element of X appears as exactly the first component of exactly one element of f.

Example. We used the notation $f(x)$ to refer to the element y such that $(x, y) \in f$ is the unique ordered pair that refers to the element $x \in X$.

Definition 2.0.8. The **Identity Function** is a function with the same domain and codomain X written $\mathbf{1}_X : X \to X$ corresponding to the diagonal 2.0.6 of X^2

Definition 2.0.9. Given $f : X \to W$ and $g : W \to Z$ with $Y \subseteq W$, the composition $g \circ f : X \to Z$ is the function satisfying $g \circ f(x) = g(f(x))$.

Definition 2.0.10. A function is **Injective** if $f(x) = f(u) \Rightarrow x = y$

Definition 2.0.11. A function $f : X \to Y$ is **Surjective** if the range of f equals Y

Definition 2.0.12. A function is **Bijective** if it is both Injective and Surjective

Theorem 2.0.1. If X is non-empty, $f : X \to Y$ is injective $\Leftrightarrow$ f is left invertible

Theorem 2.0.2. $f : X \to Y$ is surjective $\Leftrightarrow$ f is right invertible

Definition 2.0.13. A **Relation** of a set X is a subset of X^2. Conventionally written xRy rather than $(x, y) \in R$

2.0.14. Properties of Relation

1. **Reflexive** if xRx for all $x \in X$
2. **Transitive** if xRy and $yRz \Rightarrow xRz$
3. **Symmetric** if $xRy \Leftrightarrow yRx$
4. **Antisymmetric** if xRy and $yRx \Rightarrow x = y$
5. **Connex** if for every $x, y \in X$ at least on of xRy or yRx hold.

Definition 2.0.15. An **Equivalence Relation** is a relation that is Reflexive, Transitive, and Symmetric

Definition 2.0.16. if $\sim$ is an equivalence relation, the **Equivalence Class** of $x \in X$ is $[x] := \{y \in X : x \sim y\}$. Additionally, the notation $X/\sim$ refers to the set of all equivalence classes $\{[x] : x \in X\}$

2.1 The sets $\mathbb{Z}$ and $\mathbb{Q}$

Theorem 2.1.1. The natural numbers $\mathbb{N}$ with it's standard addition and multiplication is a **commutative semiring** with the following properties:

1. $(\mathbb{N}, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(\mathbb{N}, \cdot)$ Multiplication is a commutative semigroup (ie. multiplication is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 2.1.1. The integers $\mathbb{Z}$ is defined as a set of equivalence classes 2.0.16 $\mathbb{N}/\sim$ where the equivalence relation $\sim$ is

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

Remark. This is simply a rigorous way of representing integers with two natural numbers where the first natural number is the positive component and the second number is the negative component.

Theorem 2.1.2. The integers $\mathbb{Z}$ with it's standard addition and multiplication is a **commutative ring** with the following properties:

1. $(\mathbb{Z}, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(\mathbb{Z}, \cdot)$ Multiplication is a commutative semigroup (ie. multiplication is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.

Definition 2.1.2. The rational numbers $\mathbb{Q}$ is defined as a set of equivalence classes 2.0.16 $(\mathbb{Z} \times \mathbb{N})/\sim$ where the equivalence relation $\sim$ is

$$(a, n) \sim (b, m) \Leftrightarrow am = bn$$

Remark. This is simply a rigorous way of representing rational numbers as a fraction from integers and natural numbers where the integer is the numerator and the natural number is the denominator.

Theorem 2.1.3. The rational numbers $\mathbb{Q}$ with it's standard addition and multiplication is a **field** with the following properties:

1. $(\mathbb{Q}, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(\mathbb{Q}, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.
7. Multiplicative inverse.

2.1.1 Cardinality of Sets

Definition 2.1.3. The **cardinality** of a set is the number of elements in that set.

- $card(A) = card(B)$ if there exists and bijective function: $A \to B$
- $card(A) \leq card(B)$ if there exists an injective(left invertible) function: $A \to B$
- $card(A) \geq card(B)$ if there exists an surjective(right invertible) function: $A \to B$

Corollary 2.1.3.1. (Pigeonhole Principle). Suppose $n < m$ there does not exist an injective(left invertible) function: $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ and there does not exist a surjective(right invertible) function: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$

Definition 2.1.4. A set X is said to be

- **countable** if $card(X) \leq card(\mathbb{N})$
- **uncountable** if $card(X) > card(\mathbb{N})$
- **finite** if $\exists\ n \in \mathbb{N}$ such that $card(X) \leq card(\{1, \ldots, n\})$
- **countably infinite** if $card(X) = card(\mathbb{N})$
- **infinite** if $card(X) \geq card(\mathbb{N})$

Chapter 3

Order Theory

3.1 Partial Orders

Definition 3.1.1. **A Partial Order** is a relation $\preceq$ that is transitive, reflexive, and antisymmetric

Definition 3.1.2. **Poset** is a set that is equipped with a partial order.

Definition 3.1.3. Let $(X, \preceq)$ and $(U, \trianglelefteq)$ be posets we say a function $f : X \to Y$ is...

- **increasing** if $x_1 \preceq x_2 \Rightarrow f(x_1) \trianglelefteq f(x_2)$
- **decreasing** if $x_1 \preceq x_2 \Rightarrow f(x_2) \trianglelefteq f(x_1)$
- **monotone** if it is either increasing or decreasing (*note: the constant function is both increasing and decreasing*)
- **strictly increasing/decreasing/monotone** if it is increasing/decreasing/monotone and injective.
- **an order isomorphism** if it is invertible and both f and f^{-1} are increasing.

Definition 3.1.4. Let $(X, \preceq)$ and be a poset. Define the two functions $\uparrow, \downarrow : X \to 2^X$ by

- $\downarrow(x) : \{y \in X : y \preceq x\}$, a subset is a **lower set** or **downward closed** if $s \in S \Rightarrow \downarrow(s) \subseteq S$.
- $\uparrow(x) : \{y \in X : x \preceq y\}$, a subset is an **upper set** or **upper closed** if $s \in S \Rightarrow \uparrow(s) \subseteq S$

Definition 3.1.5. a lower(upper) set S is said to be **principal** if there exists $x \in X$ such that $\downarrow(x) = S(\uparrow(x) = S)$

Definition 3.1.6. Let $(X, \preceq)$ and be a poset and let $S \subseteq X$, and $z \in X$

- We say that z is an **Upper bound** of S if $S \subseteq \downarrow(z)$. The set s is said to be **bounded above** if it has an upper bound.
- We say that z is a **Lower bound** of S if $S \subseteq \uparrow(z)$. The set s is said to be **bounded bellow** if it has a lower bound.
- We say that S is order bounded (*or just bounded*), if it is bounded both above and below.

Definition 3.1.7. Let $(X, \preceq)$ and be a poset a subset $S \subseteq X$ is said to be...

- **downward directed** if every finite subset has a lower bound $z \in S$
- **upward directed** if every finite subset has a upper bound $z \in S$

3.1.1 Special Elements

Definition 3.1.8. Let $(X, \preceq)$ be a poset, and let $S \subseteq X$. We say that an element of $s_0 \in S$ is...

- **the maximum of** S if $S \subseteq \downarrow(s_0)$
- **the minimum of** S if $S \subseteq \uparrow(s_0)$
- **a maximal element of** S if, for $s \in S$, $s_0 \in \downarrow(s) \Rightarrow s_0 = s$
- **a minimal element of** S if, for $s \in S$, $s_0 \in \uparrow(s) \Rightarrow s_0 = s$

Definition 3.1.9. Let $(X, \preceq)$ be a poset and $S \subseteq X$. We say that an element of $x \in X$ is...

- **the supremum of** S if $x = \min\{y \in X : S \subseteq \downarrow(y)\}$
- **the infimum of** S if $x = \max\{y \in X : S \subseteq \uparrow(y)\}$

3.2 Total Orders

Definition 3.2.1. **A Total Order** is a relation $\preceq$ that is transitive, reflexive, antisymmetric, and connex.

Definition 3.2.2. **A Well Ordered Set** is totally ordered set where every non-empty subset has a minimum.

Definition 3.2.3. A **totally ordered field** is a field F equipped with a total order $\preceq$ such that

- $\preceq$ respects addition: $a \preceq b \Rightarrow a + c \preceq b + c$
- $\preceq$ respects positive multiplication: $0 \preceq a \Rightarrow a + c \preceq b + c$

Definition 3.2.4. In a totally ordered field, the set of **positive** elements is $\uparrow (0)\backslash 0$. The set of **negative** elements is $\downarrow (0)\backslash 0$.

Definition 3.2.5. Given a totally ordered field $(F, \preceq)$. The absolute value function $F \Rightarrow F$, denoted by $x \to |x|$, is

$$|x| = \begin{cases} x & 0 \preceq x \\ -x & x \preceq 0 \end{cases}$$

Proposition 3.2.1. Let $(X, \preceq)$ be a total order. Let A, B be both upper(both lower) sets. Either $A \subseteq B$ or $B \subseteq A$

Definition 3.2.6. Let $\{X, \preceq\}$ be a total order. We say a pair of subsets (X_-, X_+) form a **cut** of X if:

- $\{X_-, X_+\}$ is a partition of X.
- X_- is a lower set and X_+ is an upper set.

Definition 3.2.7. A totally ordered set X is said to be **Dedekind complete** if in every cut (X_-, X_+), at least one of X_- or X_+ is principal. That is $\exists x \in X_-$ such that $\downarrow (x) = X_-$ or $\exists x \in X_+$ such that $\uparrow (x) = X_+$

Proposition 3.2.2. Let $(X, \preceq)$ be a Dedekind total order. The total order restricted to $\uparrow (a)$ and $\downarrow (a)$ for any $a \in X$ is also Dedekind complete.

Definition 3.2.8. A poset $(X, \preceq)$ is said to posses the **least upper bound property** if every nonempty subset that is bounded above has a supremum. Similarly, a poset $(X, \preceq)$ is said to posses the **greatest lower bound property** if every nonempty subset that is bounded bellow has an infimum.

Theorem 3.2.1. For a totally ordered set $(X, \preceq)$

- X is a Dedekind complete
- X has the least upper bound property
- X has the greatest lower bound property

Remark. This theorem also holds for posets if you remove the Dedekind complete line since the definition of Dedekind complete relies on a total order. In general we can also define Dedekind complete as a poset that has the least upper bound property and greatest lower bound property.

Definition 3.2.9. Given $(X, \preceq)$ a partial order and elements $a, b \in X$. The **open interval** (a, b) is defined to be $\uparrow (a) \cap \downarrow (b)\backslash\{a, b\}$

Definition 3.2.10. Given $(X, \preceq)$ a partial order and elements $a, b \in X$. The **closed interval** (a, b) is defined to be $\uparrow (a) \cap \downarrow (b)$

Definition 3.2.11. Given $(X, \preceq)$ as total order with no max and no min, an **entourage mapping** is a function $f : X \to 2^X$ such that $f(x)$ is an open interval that contains x.

Definition 3.2.12. Given $(X, \preceq)$ as total order with no max and no min, we say that it possesses the **Heine-Borel property** if, for every closed interval $[a, b]$ and every entourage mapping f, there exists a finite subset $S \subseteq [a, b]$ such that $f(S)$ covers $[a, b]$.

Theorem 3.2.2. Suppose $(X, \preceq)$ as total order with no max and no min. Then it is Dedekind complete if and only if it possesses the Heine-Borel property.

3.2.1 The set $\mathbb{R}$

Definition 3.2.13. **The set** $\mathbb{R}$ is defined to be the set of all cuts (X_-, X_+) of $\mathbb{Q}$ such that X_- has no maximum.

Definition 3.2.14. We equip $\mathbb{R}$ with the relation $\leq$ defined as $(X_-, X_+) \leq (Y_-, Y_+)$ if $X_- \subseteq Y_-$.

Theorem 3.2.3. **Archimedean Property of Reals** If x, y are positive real numbers then $\exists n \in \mathbb{N}$ such that $n \cdot x > y$.

Remark. This holds for $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ as well.

Definition 3.2.15. We say a subset $U \subseteq \mathbb{R}$ is open if if for every $x \in U$, there is an open interval $(a_X, b_X) \subseteq U$ with $x \in (a_X, b_X)$

Chapter 4

Nets and Limits

Definition 4.0.1. A **directed set** is a pair $(X, \preceq)$ where X is a set equipped with a relation $\preceq$ such that

- $\preceq$ is reflexive.
- $\preceq$ is transitive.
- $\preceq$ is upward directed; for any $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

Lemma 4.0.1. Let $(X, \preceq)$ be a directed set. Let $x \in X$ the set $\uparrow(x)$ equipped with the restricted order, is a directed set. Additionally any total order is a directed set.

4.1 Nets

Definition 4.1.1. A **net** is a function $f : A \to B$ from a directed set $(A, \preceq)$.

Remark. Notationally instead of $f(a)$ we usually write f_a.

Definition 4.1.2. Given a net $f : A \to B$, we denote the **tail sets** of f by

$$f_{\uparrow(\alpha_0)} = \{f_\alpha \in B : \alpha \preceq \alpha_0\}$$

Definition 4.1.3. For a net $f : A \to B$ and a subset $S \subseteq B$ we say that:

- f is **eventually** in S if there exists some $a \in A$ such that $f_{\uparrow(a)} \subseteq S$.
- f is **frequently** in S if for every $a \in A$, the intersection $f_{\uparrow(a)} \cap S \neq \emptyset$
- f is **infrequently** in S if there exists some $a \in A$ such that $f_{\uparrow(a)}$ is disjoint from S.

4.2 Limits

Definition 4.2.1. A real valued net f is said to **Converge** to z if for every open interval I containing z, the net f is eventually in I. When f converges to z we sat that z is the **limit** of f and write $z = \lim f$.

Lemma 4.2.1. Let B be a set and $f : A \to B$ be a non-empty net. Fix a subset $S \subseteq B$.

- A net f is either frequently in S or infrequently in S.
- A net f is eventually in S if and only if f is infrequently in $X \backslash S$.
- If a net f is eventually in S, then f is frequently in S.

Definition 4.2.2. A real valued net f is said to **accumulate** (or **cluster**) at the real number z if for every open interval I containing z, the net x is frequently in I.

Proposition 4.2.1. If a real valued net f covers z, then z is its unique accumulations point.

Theorem 4.2.2. Arithmetic of Limits states that if x, y are real valued nets with the same domain then

$$\lim(x + y) = \lim(x) + \lim(y)$$

$$\lim(x \cdot y) = \lim(x) \cdot \lim(y)$$

Theorem 4.2.3. Limit Characterization of Open Sets states that a set $S \subseteq \mathbb{R}$ is open if an only if every real valued net f with an accumulation point in S is frequently in S.

Corollary 4.2.3.1. A set $S \subset \mathbb{R}$ is closed if and only if for every real valued net $f : A \to S$ has all of it's accumulation points in S.

Theorem 4.2.4. Monotone Convergence states that if f is a non-empty real valued net:

- If f is increasing and bounded above, then f converges to the supremum of its range.
- If f is decreasing and bounded below, then f converges to the infimum of its range.

Definition 4.2.3. Given a bounded, non-empty, interval $I \subseteq \mathbb{R}$, its width, which we denote by $|I|$, is a real number $|I| := \sup I - \inf I$. Necessarily $|I| \geq 0$

Definition 4.2.4. A real valued net f is a **Cauchy net** if for any positive real number ω, there exists an open interval I with a width $0 < |I| \leq \omega$ such that f is eventually in I.

Theorem 4.2.5. Cauchy's Criterion states that a real-valued net f is convergent if and only if f is Cauchy.

4.2.1 Limit superior and limit inferior

Definition 4.2.5. A real-valued net f is said to be eventually bounded (above/below) if there exists a_0 such that $f_{\uparrow(a_0)}$ is bounded (above/below)

Definition 4.2.6. Let f be a non-empty real-valued net.

- If f is eventually bounded above then its limit superior is defined as

  $$\limsup f := \lim U$$

  where $U :\uparrow (a_0) \to \mathbb{R}$ is defined as $U_a = \sup f_{\uparrow(a)}$.

- If f is eventually bounded below then its limit inferior is defined as

  $$\liminf f := \lim L$$

  where $L :\downarrow (a_0) \to \mathbb{R}$ is defined as $L_a = \inf f_{\downarrow(a)}$.

Proposition 4.2.2. Given a real-valued net f, it limit superior and limit inferior, when they exist, are the largest and smallest (respectively) accumulation points of x.

Theorem 4.2.6. Bolzano-Weierstrass Every non-empty bounded real-valued net f has an accumulation point.

Theorem 4.2.7. A real-valued net f converges if and only if it is eventually bounded and $\limsup f = \liminf f$.

Theorem 4.2.8. The Squeeze Theorem states that if x, y, z are real-valued nets with the same index set, such that both $y - x$ and $z - y$ are evntually non-negative. Then if z is eventually bounded above, and x is eventually bounded below, and $\limsup z = \liminf x = r$, then all three sequences converge to r.

Chapter 5

Metric Spaces

Definition 5.0.1. A **Metric** is a function $f : X \times X \to \mathbb{R}$ for a set X, that satisfies the following properties:

- **Positivity** - The distance $d(x_1, x_2) \geq 0$, for all $x_1, x_2 \in X$.
- **Non-degeneracy** - The distance $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$, for all $x_1, x_2 \in X$.
- **Symmetry** - The distances $d(x_1, x_2) = d(x_2, x_1)$, for all $x_1, x_2 \in X$.
- **Triangle inequality** - Given $x_1, x_2, x_3 \in X$ their mutual distances satisfy

$$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$$

Definition 5.0.2. A **Metric Space** is a set X equipped with a metric.

Proposition 5.0.1. Given a metric space (X, d) and $Y \subseteq X$, then (Y, d) is a metric space.

Definition 5.0.3. Given a metric space (X, d), an **open ball**, centered at $x \in X$ with radius r is the set $B(x, r) = \{y \in X : d(x, y) < r\}$

Definition 5.0.4. Given a metrics space (X, d), a subset $S \subseteq X$ is said to be

- **Open** if for every $x \in S$, there exists $r > 0$ such that $B(x, r) \subseteq S$.
- **Closed** if $X \backslash S$ is open

5.1 Nets in Metric spaces

Definition 5.1.1. Let (X, d) be a metric space, and let $f : A \to X$ be a net.

- We say the net f converges to $x \in X$ if f is eventually in every open ball centered as x.
- We say the net f accumulates at $x \in X$ if f is frequently in every ball centered at x.

Lemma 5.1.1. If $f : A \to X$ is a convergent net, then $\lim f$ is a unique accumulation point of X.

Theorem 5.1.2. Let (X, d) be a metric space.

1. a set $S \subseteq X$ is open $\Leftrightarrow$ every net f in X with an accumulation point in S is frequently in S.
2. a set $S \subseteq X$ is open $\Leftrightarrow$ event net f in X taking values in S has all of its accumulation points in S.

5.2 Cauchy Completeness and Compactness

Definition 5.2.1. A f taking values in a metric space (X, d) is a **Cauchy** net if, for every real number $r > 0$, there exists a point $x \in X$ such that f is eventually in $B(x, r)$.

Lemma 5.2.1. If f is a convergent net in a metric space, then f is **Cauchy**

Definition 5.2.2. A metric space (X, d) is **Cauchy Complete** if every **Cauchy** net is X converges.

Definition 5.2.3. Let (X, d) be a metric space, A subset S is **totally bounded** if, for every $r > 0$, there exists a finite subset $E \subseteq X$ such that $\{B(x, r) : x \in E\}$ covers S.

Theorem 5.2.2. Let (X, d) be a complete metric space. If S is a totally bounded subset, and f is a net that takes values in S, then f has at least one accumulation point.

Corollary 5.2.2.1. If the S from the previous theorem is also closed, then f has at least one accumulation point in S.

Definition 5.2.4. Let (X, d) be a complete metric space. A subset $K \subseteq X$ is said to be compact if every net taking values in K has an accumulation point in K

Theorem 5.2.3. Let (X, d) be a complete metric space. The following statements about a subset K are equivalent.

1. K is compact.

2. K is closed and totally bounded.

3. $\mathscr{S}$ is a collection of open subsets of X and $\mathscr{S}$ covers K, then there exists a finite subset $\mathscr{D} \subseteq \mathscr{S}$ that also covers K.

4. If $f : K \to \mathbb{R}$ is a positive function, then there exists a finite subset $S \subseteq K$ such that $\{B(x, f(x)) : x \in S\}$ covers K.

Chapter 6

Subnets and Infinite sums

6.1 Subnets

Definition 6.1.1. Given a net $x : A \to X$, a net $y : B \to X$ is said to be a subnet of x, if there exists a function $\varphi : B \to A$ such that

- $y = x \circ \varphi$
- φ is increasing: $b_1 \leq b_2 \Rightarrow \varphi(b_1) \leq \varphi(b_2)$
- For every $a \in A$, there exists $b \in B$ with $\varphi(b) \geq a$.

Proposition 6.1.1. If y is a subnet of x, and z is a subnet of y, then z is a subnet of x.

Proposition 6.1.2. Let x be a net in X. Suppose $S \subseteq X$ is such that x is frequently in S. Then there exists a subnet y of x that is eventually in S.

Theorem 6.1.1. If x is a net in X, and y is a subnet.

- If x is eventually in S, then y is eventually in S.
- If y is frequently in S, then x is frequently in S.

Corollary 6.1.1.1. Let x be a net and y be a subnet. Then if x converges to a point z, then so does y. If y accumulates at a point z' then so does x.

Theorem 6.1.2. Let X be the real line or a metric space. Let $x : A \to B$ be a net, and z an accumulation points of x. Then there exists a subnet y of x that converges to z.

Corollary 6.1.2.1. Let x be a net, then z is an accumulation points of x if and only if there exists a subnet y that converges to z

Theorem 6.1.3. Let x be a net, then the set of all of its accumulation points is a closed set.

6.2 Infinite sums

Let $\mathcal{I}$ be an arbitrary set. Consider a function $\tau : \mathcal{I} \to \mathbb{R}$. Let A be the set of all finite subsets of $\mathcal{I}$ ordered by inclusion. A is a subset of the poset $2^{\mathcal{I}}$ and hence is a poset, it is also directed since if α_1, α_2 are two finite subsets of $\mathcal{I}$, so is the set $\alpha_1 \cup \alpha_2$ which succeeds both α_1 and α_2. Now we can construct a net $x : A \to \mathbb{R}$ where

$$x_a = \sum_{i \in \alpha} \tau(i)$$

This is well defined since α is finite and real numbers are closed under addition. We can interpret the limits of the net x, if it exists as the infinite sum of τ.

Proposition 6.2.1. If $\tau(u) \neq 0$ for uncountably many, then the net x cannot converge.

Assumption 6.2.1. For this section we will assume we are adding a countably infinite list of non-zero numbers. We will fix the following notations.

- We let $\mathcal{I}$ be a countably infinite set, and $\tau : \mathcal{I} \to \mathbb{R}\backslash\{0\}$ be the list on non-vanishing terms.
- We will denote by $A \subseteq 2^{\mathcal{I}}$ the set of all finite subsets of natural numbers.
- We have the net $x : A \to \mathbb{R}$, where $x_\alpha = \sum_{i \in \alpha} \tau(i)$ is the infinite sum.

Definition 6.2.1. Given a particular enumeration $\eta : \mathbb{N} \to \mathcal{I}$, the **associated series** is the sequence σ where $\sigma_n = \sum_{j=1}^{n} \tau(\eta(j))$. The series is said to converge if the sequence σ converges, in which case we write $\sum_\eta \tau = \lim \sigma$.

Lemma 6.2.1. σ is a subnet of x.

6.2.1 Absolute convergence

Definition 6.2.2. We say that the infinite sum of τ converges absolutely if the net x converges. In this case we write $\sum_{abs} \tau = \lim x$.

Lemma 6.2.2. If there exists an enumeration n such that the corresponding $\sum_n \tau$ converges, then for every $r > 0$, the set $\{i \in \mathcal{I} : |\tau(i)| \geq r\}$ is finite.

Theorem 6.2.3. Suppose there exists and enumeration η for which $\sum_\eta \tau$ converges, and suppose that exactly one of $\mathcal{I}^{+-}$ is finite, then the infinite sum of τ converges absolutely and $\sum_{abs} \tau = \sum_\eta \tau$.

Theorem 6.2.4. Let τ take only positive values, and suppose $\sum_{abs} \tau$ converges absolutely. If $\mu : \mathcal{I} \to \mathbb{R}$ is any function such that for every i, $|\mu(i)| \leq \tau(i)$, then the infinite sum of μ also converges absolutely.

Theorem 6.2.5. Denote by τ^+ the restriction of τ to $\mathcal{I}^+$ and τ^- the restriction of τ to $\mathcal{I}^-$. Suppose the sums $\sum_{abs} \tau^+$ and $\sum_{abs} \tau^-$ both converge absolutely, then the infinite sum for τ converges absolutely and $\sum_{abs} \tau = \sum_{abs} \tau^+ + \sum_{abs} \tau^-$.

6.2.2 Conditional convergence

Definition 6.2.3. We say the infinite sum of τ converges conditionally if the net x does not converge and for some enumeration η the sum $\sum_\eta \tau$ converges.

Lemma 6.2.6. Suppose x is divergent but there exists an enumeration η such that the series σ converges. Then there exists enumerations of v_{+-} of $\mathcal{I}^{+-}$ respectively, such that $\tau \circ v_+$ is a decreasing function, and $\tau \circ v_-$ is an increasing function.

Theorem 6.2.7. Riemann rearrangement theorem states that if x is divergence and there exists an enumeration η_0 such that the corresponding series converges. Let $x \in \mathbb{R}$. Then there exists a possibly different enumeration of η of $\mathcal{I}$ such that the corresponding series converges, with $\sum_\eta \tau = z$.

Theorem 6.2.8. If there exists an enumeration η whose series converges, then the net x corresponding to the infinite sum of τ satisfies exatctly on of the following:

- x converges; or
- the set of all accumulation points of x is $\mathbb{R}$.

Chapter 7

Continuity

7.1 Continuity

Definition 7.1.1. Let $S \subseteq \mathbb{R}$, and let $z \in S$. A function $f : S \to$ is said to be continuous at the point z if for every real valued net x, which takes values only in S, and which converges to z we have $\lim f \circ x = f(\lim x) = f(z)$.

Theorem 7.1.1. A function $f : S \to \mathbb{R}$ is continuous at $z \in S$ if and only if for every open interval $J \ni f(z)$ there exists an open interval $I \ni z$ such that $f(S \cup I) \subseteq J$

Theorem 7.1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous on X if and only if the induced power set function $f^{-1} : 2^Y \to 2^X$ maps closed subsets to closed subsets.

Lemma 7.1.3. If $f : X \to Y$ is any function. then the induced power set map $f^{-1} : 2^Y \to 2^X$ maps closed subsets to closed subsets if and only if it maps open sets to open sets.

7.1.1 Other modes of continuity

Definition 7.1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. a function $f : X \to Y$ is said to be uniformly continuous if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $x, x' \in X$, the bound $d_X(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \epsilon$.

Definition 7.1.3. Let (X, d_X) and (Y, d_Y) be metric spaces. a function $f : X \to Y$ is said to have a removable discontinuity at x_0 if f is discontinuous at x_0 but there exists $g : X \to Y$ that is continuous at x_0, such that $f(x) = g(x)$ for all $x \neq x_0$.

Proposition 7.1.1. A function f has a removable discontinuity or is continuous at x_0 if any only if for every net $\mu \to x_0$, with μ taking values only in $X \backslash \{x_0\}$, the net $f \circ \mu$ converges in Y.

Definition 7.1.4. Let (X, d_X) be a metric space, and $f : X \to \mathbb{R}$ a function.

- f is said to be upper semi-continuous at a point x_0 if for every net $\mu \to x_0$, its value $f(x_0) \geq \limsup f \circ \mu$
- f is said to be lower semi-continuous at a point x_0 if for every net $\mu \to x_0$, its value $f(x_0) \leq \limsup f \circ \mu$

Proposition 7.1.2. A real-valued function f is continuous at x_0 if and only if it is both upper semi-continuous and lower semi-continuous at x_0.

Theorem 7.1.4. Let (X, d_X) be a metric space, and $f : X \to \mathbb{R}$ a function. Denote by $f^{-1} : 2^{\mathbb{R}} \to 2^X$ its induced power set mapping.

- f is upper semi-continuous on all of X if and only if $f^{-1}(\uparrow (y))$ is closed for every $y \in \mathbb{R}$.
- f is lower semi-continuous on all of X if and only if $f^{-1}(\downarrow (y))$ is closed for every $y \in \mathbb{R}$.

7.2 Interpolation and Extrapolation

7.2.1 Intermediate Value Theorem

Theorem 7.2.1. Intermediate Value - Let $f : [a, b] \to \mathbb{R}$ be a continuous function, Suppose $f(a) \neq f(b)$, then for any γ strictly between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ such that $f(c) = \gamma$.

Definition 7.2.1. a **Darboux function** is a function with the intermediate value property.

Definition 7.2.2. A metric space (X, d) is said to be connected if there does not exist a partition $\{X_1, X_2\}$ of X into closed subsets. The metric space is said to be disconnected otherwise.

Theorem 7.2.2. Suppose (X, d_X) is a connected metric space, and (Y, d_Y) is disconnected, with partition $\{Y_1, Y_2\}$ by closed subsets. if $f : X \to Y$ is a continuous function, then $f(X)$ then $f(X)$ can only intersect one of Y_1 and Y_2.

7.2.2 Dense subsets

Definition 7.2.3. A subset S of a metric space (X, d) is said to be dense if for every $x \in X$ and $r > 0$, $B(x, r) \cup S \neq \emptyset$.

Proposition 7.2.1. Let f, g be continuous function from a metric space (X, d_X) to (Y, d_Y). suppose the restriction of f and g to a dense subset are equal, then $f = g$ everywhere.

Theorem 7.2.3. Let (X, d_X) and (Y, d_Y) be metric space, with Y Cauchy . Let $S \subseteq X$ be a dense subset; note that d_X restricts to a metric space on S. Given a uniformly continuous function $f : S \to Y$, then there exists a unique uniformly continuous function $f : X \to Y$ that extends f.

Lemma 7.2.4. If (X, d_X) and (Y, d_Y) are metric space, and $f : X \to Y$ is uniformly continuous, then for any Cauchy net μ in X, the net $f \circ \mu$ is Cauchy.

7.3 Continuous functions and compact sets

Theorem 7.3.1. Let $f : K \to Y$ where K is a compact subset of a Cauchy complete metric space, and Y is a metric space. If f is continuous, then f is uniformly continuous.

Theorem 7.3.2. Let $f : X \to Y$ be a continuous function between two Cacuhy complete metric spaces. If $K \subseteq X$ is compact then $f(K)$ is also compact.

Theorem 7.3.3. Let $f : K \to Y$ be a continuous bijection, where K is compact and complete, and Y a Cauchy complete metric space. Then the inverse function f^{-1} is also continuous.

Lemma 7.3.4. If K is a compact metric space, and $F \subseteq K$ is closed, then F is also compact.

Theorem 7.3.5. Extremal Value Theorem. Let $f : K \to \mathbb{R}$ be continuous, where K is a compact subset of a metric space. Then $\sup f(K) \in f(K)$ and $\inf f(K) \in f(K)$.

Lemma 7.3.6. For any compact $C \subseteq \mathbb{R}$, both $\sup C$ and $\inf C$ are elements of C.

Corollary 7.3.6.1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then the range of f is a closed interval.

Chapter 8

Differentiability

8.1 Differentiability

8.1.1 Tangency

Definition 8.1.1. (o notation) Let (X, d) be a metric space, $x \in X$ a point, and $\alpha > 0$ a real number. We say that a function $f : X \to \mathbb{R}$ is in the set $o(x, \alpha)$ if, for every $\epsilon > 0$, there exists $r > 0$ such that restricted to $B(x, r)$, the function f satisfies $|f(y)| < \epsilon \cdot d(x, y)^\alpha$.

Proposition 8.1.1. If $f \in o(x, \alpha)$, then for every net $\mu \to x$ taking values in $X \backslash \{x\}$ we have that $\lim f \circ \mu \cdot d(x, \mu)^{-\alpha} = 0$.

Definition 8.1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Two function f and g, both from X to Y, are said to be **tangent** at $x_0 \in X$, if the function $X \ni x \to d(f(x), g(x)) \in \mathbb{R}$ is an element of $o(x_0, 1)$.

8.1.2 Definition of Differentiability

Definition 8.1.3. The set of **affine functions** on $\mathbb{R}$, denoted Aff is the set of all function of the form $x \mapsto m \cdot x + b$ for $m, b \in \mathbb{R}$. The number m we refer to as the slope of the affine function.

Definition 8.1.4. Let $S \subseteq \mathbb{R}$. A function $f : S \to \mathbb{R}$ is said to be **differentiable** at the point $x_0 \in S$ if it is tangent at x_0 to an element of Aff.

Proposition 8.1.2. if $x_0 \in S$ is not isolated. Then at most one element of Aff can be tangent to any given function f at x_0.

Definition 8.1.5. Let $S \subseteq \mathbb{R}$. Suppose $x_0 \in S$ is not an isolated point and $f : S \to \mathbb{R}$ is differentiable at x_0. By the **derivative** of f at x_0 we refer to the function $f' : S \to \mathbb{R}$ such that $f'(x)$ is the slope of the unique element of Aff that is tangent to f at x_0.

Theorem 8.1.1. (Sum Rule and Product Rule) Let $S \subseteq \mathbb{R}$ and f, g are function $S \to \mathbb{R}$. Suppose $x_0 \in S$ is not isolated, and f and g are both differentiable at x_0 with derivatives m_f and m_g respectively. Then:

1. The function $h = f + g$ is differentiable at x_0 with derivative $m_f + m_g$.
2. The function $h = f \cdot g$ is differentiable at x_0 with derivative $m_f g(x_0) \cdot m_g f(x_0)$.

Theorem 8.1.2. (Chain Rule) Let $S \subseteq \mathbb{R}$ with x_0 a non-isolated point in S. Suppose $f : S \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is such that f is differentiable at x_0 with derivative m_f and g is differentiable at $f(x_0)$ with derivative m_g. Then $g \circ f$ is differentiable at x_0 with derivative $m_g m_f$

Proposition 8.1.3. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ with $f'(x) \equiv m$. Then $f \in \mathsf{Aff}$.

Definition 8.1.6. Let $S \subseteq \mathbb{R}$ be a set with no isolated poitns. A function $f : S \to \mathbb{R}$ is said to be **continuously differentiable** if f is differentiable at every point in S, and the function $f' : S \to \mathbb{R}$ is continuous. This property is sometimes written $f \in \mathcal{C}^1(S; \mathbb{R})$

8.2 Mean Value Theorems and Applications

Lemma 8.2.1. (Fermat's stationary point lemma). Let $f : (a, b) \to \mathbb{R}$ such that f is differentiable at some $c \in (a, b)$ and $f(c) \geq f(x)$ for any $x \in (a, b)$. Then f has a derivative of 0 at c.

Proposition 8.2.1. Let f, g be continuous functions on $[a, b]$ and differentiable on (a, b), such that $f(a) = g(a)$ and $f(b) = g(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = g'(c)$.

8.2.1 Mean Value Theorems

Theorem 8.2.2. (Cauchy's Mean Value Theorem). Let f, g be continuous functions from $[a, b] \to \mathbb{R}$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c)[g(a) - g(b)] = g'(c)[f(a) - f(b)]$$

Corollary 8.2.2.1. (Mean Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose f is differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = [f(a) - f(b)]/(a - b)$

Corollary 8.2.2.2. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function then

1. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b).
2. If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

Definition 8.2.1. (Lipschitz continuous). A function is said to be **Lipschitz continuous** (denoted $f \in \mathcal{C}^{0,1}(X, Y)$) if there exists some M such that $d_Y(f(x), f(x')) \leq M d_X(x, x')$ for all $x, x' \in X$. The infimum of all values M for which the inequality holds is called the **Lipschitz constant** of the function f.

Proposition 8.2.2. if f is Lipschitz continuous, then f is uniformly continuous.

Theorem 8.2.3. Let I be an interval. if $f : I \to \mathbb{R}$ is differentiable on I, such that $|f'| \leq M$ on I, then f is continuous with Lipschitz constant at most M.

8.2.2 Darboux's Theorem and Consequences

Theorem 8.2.4. (Darboux's Theorem). Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$. Then the derivative f' is a Darboux function.

Corollary 8.2.4.1. Let I be an interval, and $f : I \to \mathbb{R}$ real valued function differentiable on I. Suppose $f'(x) \neq 0$ for all $x \in I$, then f is strictly monotonic on I.

Theorem 8.2.5. (Inverse function theorem). Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, such that $f'(x) \neq 0$ for any $x \in [a, b]$. Then

- There exists $g : f([a, b]) \to [a, b]$ such that $g \circ f$ is the identity map.
- The function g is differentiable on its domain with

$$g'(y) = \frac{1}{f'(g(y))}$$

8.2.3 L'Hopital's Rule

Theorem 8.2.6. (L'Hopital's Rule). Let $I = (a, b)$ or $\uparrow (a)$, considered as a directed set with the usual ordering. Let f, g be functions mapping $I \to \mathbb{R}$, such that they are both differentiable on I. Assume:

- $g'(x) \neq 0$, for all $x \in I$;
- as nets, $\lim f = 0 = \lim g$;
- the net $\lim(f'/g') = \alpha$.

Then the net f/g is well defined, and has limit $\lim f/g = \alpha$.

8.3 Second Derivatives

Theorem 8.3.1. Suppose $f \in \mathcal{C}^1((a,b);\mathbb{R})$, such that f' is differentiable at $c \in (a,b)$ with derivative m_2. Then the function

$$x \mapsto f(x) - f(c) - f'(c)(x-c) - \frac{1}{2}m_2(x-c)^2$$

is in $o(c,2)$

Theorem 8.3.2. Suppose $f \in \mathcal{C}^1((a,b),\mathbb{R})$, such that f' is differentiable on (a,b). Denote by f'' the derivative of f'. Then for $c,d \in (a,b)$, there exists s between c and d such that

$$f(d) = f(c) + f'(c)(d-c) + \frac{f''(s)}{2}(d-c)^2$$

Chapter 9

Riemann Integral

9.1 Riemann Integral

Definition 9.1.1. Given a closed bounded interval $[a, b] \subsetneq \mathbb{R}$:

- A **tagged subinterval** is an ordered pair (τ, I) where $I \subseteq [a, b]$ is a closed interval and $\tau \in I$ is the **tag** of the subinterval I.
- A **tagged division** of $[a, b]$ is a finite set $\mathcal{T}$ of tagged subintervals such that
  - the union of all the sub intervals appearing in $\mathcal{T}$ equals $[a, b]$.
  - the sum of the widths of all the subintervals appearing in $\mathcal{T}$ equals $b - a$.

Definition 9.1.2. Given $f : [a, b] \to \mathbb{R}$, and a tagged division $\mathcal{T}$ of $[a, b]$ the corresponding **Riemann sum** of f is

$$S_{\mathcal{T}} f := \sum_{(\tau,I)\in\mathcal{T}} f(\tau) \cdot |I|$$

Definition 9.1.3. The **width of a tagged division** $\mathcal{T}$ is the width of it's widest subinterval.

Proposition 9.1.1. Denote the set of all tagged divisions with a width less than δ:

$$r([a, b]) := \{(\delta, \mathcal{T}) : \delta > |\mathcal{T}| > 0\}$$

This set is a directed set.

Definition 9.1.4. A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** (abbreviated $f \in \mathcal{R}([a, b])$), if the net $\rho[f] : r([a, b]) \to \mathbb{R}$ given by $\rho[f]_{(\delta,\mathcal{T})} = \mathcal{S}_{\mathcal{T}} f$ converges. We denote the value of the limit by $\int_a^b f(x)dx$.

9.1.1 Some technical lemmas

Definition 9.1.5. A tagged division $\mathcal{T}'$ is said to be a **refinement** of $\mathcal{T}$ if for each subinterval I' that appears in $\mathcal{T}'$, there exists a subinterval I of $\mathcal{T}$ such that $I' \subseteq I$.

Definition 9.1.6. Given a tagged division $\mathcal{T}$ of an interval $[a, b]$ and a bounded function $f : [a, b] \to \mathbb{R}$ the **upper and lower Darboux sums** corresponding to $\mathcal{T}$ are:

$$\overline{S}_{\mathcal{T}} f = \sum_{(\tau,I)\in\mathcal{T}} (\sup f(I)) \cdot |I|$$

$$\underline{S}_{\mathcal{T}} f = \sum_{(\tau,I)\in\mathcal{T}} (\inf f(I)) \cdot |I|$$

Lemma 9.1.1. if $\mathcal{T}'$ is a refinement of $\mathcal{T}$, and f is a bounded function, then

$$\underline{S}_{\mathcal{T}} f \leq \underline{S}_{\mathcal{T}'} \leq \overline{S}_{\mathcal{T}'} f \leq \overline{S}_{\mathcal{T}}$$

Lemma 9.1.2. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tagged divisions, then there exists a tagged division $\mathcal{T}$ that is a refinement of both $\mathcal{T}_1$ and $\mathcal{T}_\in$

Lemma 9.1.3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tagged divisions, and f a bounded function, then

$$|S_{\mathcal{T}_1} f - S_{\mathcal{T}_2} f| \leq \overline{S}_{\mathcal{T}_\infty} f - \underline{S}_{\mathcal{T}_\infty} f + \overline{S}_{\mathcal{T}_\in} f - \underline{S}_{\mathcal{T}_\in f}$$

Theorem 9.1.4. (**Darboux's Criterion**) A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every tagged division $\mathcal{T}$ with a width less than δ

9.2 Properties of the Riemann Integral

Theorem 9.2.1. If $f : [a, b] \to \mathbb{R}$ is continuous, the $f \in \mathcal{R}([a, b])$.

Definition 9.2.1. A subset $s \subseteq \mathbb{R}$ is a **null** set if for every $\epsilon > 0$, there exists a countable collection of $\mathscr{U}$ of open intervals such that

1. $\cup\mathscr{U} \supseteq S$
2. the (possibly infinite) sum $\sum_{I\in\mathscr{U}} |I|$ converges (absolutely) to be less than ϵ.

Proposition 9.2.1. Any countable set is a null set.

Theorem 9.2.2. (**Lebesgue Criterion for Riemann Integrability**) A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of points on which it is discontinuous is a null set.

Proposition 9.2.2. If $f : [a, b] \to [m, M]$ is Riemann integrable, and $g : [m, M] \to \mathbb{R}$ is continuous, then $g \circ f \in \mathcal{R}([a, b])$.

Proposition 9.2.3. (Properties of Integrals)

1. If $f, g \in \mathcal{R}([a, b])$, and $c, d \in \mathbb{R}$, then $cf + dg \in \mathcal{R}([a, b])$ and $\int_a^b cf + dgdx = c\int_a^b fdx + d\int_a^b gdx$.
2. If $f, g \in \mathcal{R}([a, b])$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b fdx \leq \int_a^b gdx$.

Proposition 9.2.4. Given an interval $[a, b]$ and $c \in (a, b)$. Let $f : [a, b] \to \mathbb{R}$, and write f_1 for its restriction to $[a, c]$ and f_2 for its restriction to $[c, b]$. Then $f \in \mathcal{R}([a, b])$ if and only if $f_1 \in \mathcal{R}([a, c])$ and $f_2 \in \mathcal{R}([c, b])$.

9.3 Lebesgue Criterion

Definition 9.3.1. Given $f : [a, b] \to \mathbb{R}$ a bounded function, define the function $osc : 2^{[a,b]} \to \mathbb{R}$ given by

$$\mathbf{osc}(S) = \sup f(S) - \inf f(S)$$

Given $x \in [a, b]$, and $\mathscr{I}_x$ the set of open intervals containing x ordered by inclusion. We can define the net:

$$\omega_f(x) := \lim(\mathscr{I}_x \ni I \mapsto osc(I \cap [a, b]))$$

Lemma 9.3.1. The function f is continuous at x if and only if $\omega_f(x) = 0$

Let

$$D_k := \{x \in [a, b] : \omega_f(x) \geq \frac{1}{k}\}$$

$$D := \bigcup\{D_k : k \in \mathbb{N}\}$$

Lemma 9.3.2. D is a null set if and only if for every $k \in \mathbb{N}$, the set D_k is a null set.

Lemma 9.3.3. D_k is compact.

Theorem 9.3.4. (Lebesque Criterion) The function f is Riemann Integrable if and only if D is a null set. Where D is the set of all discontinuities.

9.4 Indefinite Integrals and Derivatives

Definition 9.4.1. It is often valuable to consider reverse integrals that is taking across enpoints in the wrong order. For these cases we will accept the following convention:

$$\text{if } a > b, \text{ then } \int_a^b f(x)dx = -\int_b^a f(x)dx$$

Definition 9.4.2. Let $f \in \mathcal{R}([a, b])$, and $c \in [a, b]$, the **indefinite Riemann** integral of f based at c is the function $F : [a, b] \to \mathbb{R}$ given by $F(x) = \int_c^x f(y)dy$.

Proposition 9.4.1. Let $f \in \mathcal{R}([a, b])$ and $c \in [a, b]$, then $\int_a^c f(x)dx = \int_c^b f(x)dx = \int_a^b f(x)dx$.

Proposition 9.4.2. If $f \in \mathcal{R}([a, b])$, and F its indefinite Riemann integral based at come $c \in [a, b]$, then F is uniformly Lipschitz continuous.

9.5 Fundamental Theorems of Calculus

Theorem 9.5.1. (Fundamental Theorem of Calculus: derivatives of integrals) Let $f \in \mathcal{R}([a,b])$ and F an indefinite Riemann integral of f. if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Corollary 9.5.1.1. (Existence of anti-derivatives). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then there exists $F \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ such that $F' = f$.

Corollary 9.5.1.2. If $f \in \mathcal{R}([a,b])$ and $f(x) > 0$ for all x, then $\int_a^b f(x)dx > 0$.

Theorem 9.5.2. (Fundamental Theorem of Calculus: integrals of derivatives) Let $f : [a,b] \to \mathbb{R}$ be differentiable. If $f' \in \mathcal{R}([a,b])$ then for every $x, y \in [a,b]$, $f(y) - f(x) = \int_x^y f'(s)ds$.

9.5.1 Integration By Parts

Theorem 9.5.3. Let $f, g \in \mathcal{R}([a,b])$, and denote by $F(x) = \int_a^x f(y)dy$ and $G(x) = \int_a^x g(y)dy$. Then

$$F(b)G(b) = \int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx$$

Chapter 10

Henstock and Stieltjes Integral

10.1 Henstock Integral

Definition 10.1.1. If $\eta : [a,b] \to 2^{\mathbb{R}}$ is an entourage mapping. we say that the tagged division $\mathcal{T}$ is η-fine if, for every $(\tau, I) \in \mathcal{T}$, we have $U \notin \eta(\tau)$.
We can expand this definition with the ordering

$$(\eta, \mathcal{T}) \preceq (\eta', \mathcal{T}') \Leftrightarrow [\forall x \in [a,b] : n'(x) \subseteq \eta(x)].$$

This defines the following directed set

$$h([a,b]) := \{(\eta, \mathcal{T}) : \eta \text{ is an entourage mapping}, \mathcal{T} \text{ is } \eta\text{-fine}\}$$

Now we can define the net $\varsigma[f] : h([a,b]) \to \mathbb{R}$ by

$$\varsigma[f]_{\eta,\mathcal{T}} = S_{\mathcal{T}} f$$

Proposition 10.1.1. $\varsigma[f]$ is a subnet of $\rho[f]$.

Definition 10.1.2. We say the function $f : [a,b] \to \mathbb{R}$ is Henstock Integrable, written $f \in \mathcal{H}([a,b])$, if the net $\varsigma[f]$ converges. We will write using $\int_a^b f(x)dx = \lim \varsigma[f]$.

Proposition 10.1.2. If $f : [a,b] \to \mathbb{R}$ is such that the set $\{x \in [a,b] : f(x) \neq 0\}$ is a null set, then $f \in \mathcal{H}([a,b])$ with integral 0.

10.1.1 Properties of Henstock Integrals

Proposition 10.1.3.

1. If $f, g \in \mathcal{H}([a,b])$, and $c, d \in \mathbb{R}$, then $cf + dg \in \mathcal{R}([a,b])$ and $\int_a^b cf + dg dx = c\int_a^b f dx + d\int_a^b g dx$.
2. If $f, g \in \mathcal{R}([a,b])$ and $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f dx \leq \int_a^b g dx$.
3. If $f \in \mathcal{H}([a,b])$, and g differs from f only on a null subset of $[a,b]$, then $g \in \mathcal{H}([a,b])$ and $\int_a^b f(x)dx = \int_a^b g(x)dx$.

Theorem 10.1.1. (Fundamental Theorem of Calculus for Henstock Integrals) Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Then $f' \in \mathcal{H}([a,b])$ and $\int_a^b f'(x)dx = f(b) - f(a)$.

Lemma 10.1.2. Let $f : [a,b] \to \mathbb{R}$, and take $c \in [a,b]$. Suppose $f \in \mathcal{H}([a,b])$. Then

$$f \in \mathcal{H}([a,b]) \Leftrightarrow f \in \mathcal{H}([c,b])$$

When either hold, we have $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Theorem 10.1.3. (Hake's Theorem) Given $f : [a,b] \to \mathbb{R}$ such that $f \in \mathcal{H}([a,c])$ for every $c \in [a,b)$ and such that $\lim_{c \to b^-} \int_a^c f(f)dx = L$ converges, then $f \in \mathcal{H}([a,b])$ with $\int_a^b f(x)dx = L$.

10.2 Stieltjes Integral

Definition 10.2.1. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an increasing function and let $\mathcal{T}$ be a tagged division of some closed, bounded interval $[a, b]$. Given a function $f : [a, b] \to \mathbb{R}$, the **Riemann-Stieltjes** sum with a weight α with respect to the tagged division $\mathcal{T}$ is

$$S^{\alpha}_{\mathcal{T}} f := \sum_{(\tau,I)\in\mathcal{T}} f(\tau)(\alpha(\sup I) - \alpha(\inf I))$$

We can analogously define the net $\rho[f]^{(\alpha)} : r([a, b]) \to \mathbb{R}$ and $\varsigma[f]^{(\alpha)} : h([a, b]) \to \mathbb{R}$ using the Stieltjes sum in place of the Riemann sum. To differentiate the Stieltjes integrals from there Riemann/Henstock counterparts, we will denote the limits of $\varsigma[f]^{(\alpha)}$ and $\rho[f]^{(\alpha)}$, with the notation

$$\int_a^b f(x)d\alpha$$

Proposition 10.2.1. For any monotonic weight α, $\varsigma[f]^{(\alpha)}$ is a subnet of $\rho[f]^{(\alpha)}$ and the properties of Riemann integrals and Henstock integrals follow for the corresponding Stieltjes integrals.

Proposition 10.2.2. If α, β are two increasing functions on $\mathbb{R}$, and $f \in \mathcal{R}([a, b], \alpha) \cap \mathcal{R}([a, b], \beta)$, then $f \in \mathcal{R}([a, b], \alpha + \beta)$ with $\int_a^b f(x)d(\alpha + \beta) = \int_a^b f(x)d\alpha + \int_a^b f(x)d\beta$ that same is true for Henstock-Stieltjes integrals.

10.3 Properties of the Riemann-Stieltjes Integral

Theorem 10.3.1. If f is continuous on $[a, b]$, then $f \in \mathbb{R}([a, b], \alpha)$ for any increasing function α.

Proposition 10.3.1. If there exists $c \in [a, b]$ such that both α and f are discontinuous at c, then $f \notin \mathcal{R}([a, b], \alpha)$.

Proposition 10.3.2. Let α be an increasing function such that $\alpha(x) = 0$ when $x < 0$ and $\alpha(x) = 1$ when $x > 0$. Then for any $a < 0 < b$ and $f : [a, b] \to$ that is continuous at 0 we have

$$\int_a^b f(x)d\alpha = f(0)$$

10.4 Change of variables and integration by parts

Theorem 10.4.1. Let $\alpha : [a, b] \to \mathbb{R}$ be increasing and continuous and $f : [c, d] \to \mathbb{R}$, where the interval $[c, d] = \alpha([a, b])$. if $f \in \mathcal{R}([c, d])$, then $f \circ \alpha \in \mathcal{R}([a, b], \alpha)$ with

$$\int_a^b f(\alpha)d\alpha = \int_c^d f(x)dx$$

This theorem also holds for Henstock-Stieltjes integrals.

Theorem 10.4.2. Let $\alpha : [a, b] \to \mathbb{R}$ be increasing and differentiable, with $\alpha' \in \mathcal{R}([a, b])$; and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}([a, b], \alpha)$ if and only if $f \cdot \alpha' \in \mathcal{R}([a, b])$; when this holds the integrals

$$\int_a^b f(x)\alpha'(x)dx = \int_a^b f(x)d\alpha$$

Corollary 10.4.2.1. (Change of variables) Let $u \in \mathcal{C}^1([a, b]; \mathbb{R})$, with $u'(x) > 0$ for all $x \in [a, b]$. Then

$$f \in \mathcal{R}(u[a, b]) \Leftrightarrow f(u) \cdot u' \in \mathcal{R}([a, b])$$

$$\int_{u(a)}^{u(b)} f(x)dx = \int_a^b f(u(x)) \cdot u'(x)dx$$

Theorem 10.4.3. If $\mu, v : [a, b] \to \mathbb{R}$ are both continuous and increasing, then

$$\mu(b)v(b) - \mu(a)v(a) = \int_a^b \mu(x)dv + \int_a^b v(x)dv$$