Topology

from the context of the course
MTH 461: Metric and Topological Spaces

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Contents

Chapter 1

Fundamentals

1.1 Functions

Definition 1.1.1. A **function** $f : A \to B$ is a subset of $X \times Y$ such that $\forall x \in X$, $\exists$ exactly one element $y \in B$, $(x, y) \in f$.

Definition 1.1.2. The **domain** of a function $f : A \to B$ is $\{a \in A : \exists b \in B \text{ such that } (a, b) \in f\}$.

Definition 1.1.3. The **range** of a function $f : A \to B$ is $\{b \in B : \exists a \in A \text{ such that } (a, b) \in f\}$.

Definition 1.1.4. A function is a **injective** denoted $f : A \hookrightarrow B$ iff $f(x) = f(u) \Rightarrow x = y$.

Definition 1.1.5. A function is a **surjection** denoted $f : A \twoheadrightarrow B$ iff the range of f equals B.

Definition 1.1.6. A function is a **bijection** denoted $f : A \hookrightarrow\!\!\!\!\rightarrow B$ iff it is both an injection and a surjection.

1.2 Relations

Definition 1.2.1. A **relation** on a set A is a subset of $A \times A$. Conventionally written xRy rather than $(x, y) \in R$.

Definition 1.2.2. For a relation R on a set A, R is

- **Reflexive** iff xAx for all $x \in A$
- **Antireflexive** iff $\nexists x \in A$ such that xAx
- **Transitive** iff xRy and $yRz \Rightarrow xRz$, for any $x, y, z \in A$.
- **Symmetric** iff $xRy \Leftrightarrow yRx$, for any $x, y \in A$.
- **Antisymmetric** iff xRy and $yRx \Rightarrow x = y$, for any $x, y \in A$.
- **Connex** iff for every $x, y \in R$ at least on of xRy, yRx, or $x = y$ hold.

Definition 1.2.3. An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

Definition 1.2.4. The **equivalence class** of $a \in A$ for a relation $\sim$ is $[x] := \{b \in A | a \sim b\}$.

Definition 1.2.5. A **partition** of a set A is a set of subsets X such that $\bigcup X = A$ and $\forall B, C \in X, A \neq B \Rightarrow A \cap B = \emptyset$.

Lemma 1.2.1. Let $x, y \in A$ and $\sim$ be an equivalence class on A, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

Corollary 1.2.1.1. Any partition defines and equivalence relation and vice versa.

1.3 Order

Definition 1.3.1. An **order** on a set A is a relation that is antireflexive, transitive, and connex.

Definition 1.3.2. A **partial order** on a set A is a relation that is reflexive, antisymmetric, and transitive.

Definition 1.3.3. Two ordered sets have the same **order type** if there exists a bijection that preserves order.

Definition 1.3.4. Let $(X, \leq)$ be an ordered set, and let $A \subseteq X$.

- The **maximum** of A is an element $a_{max} \in A$ such that $\forall a \in A, a \leq a_{max}$.
- The **minimum** of A is an element $a_{min} \in A$ such that $\forall a \in A, a \geq a_{min}$.
- An **upper bound** of A is an element $x \in X$ such that $\forall a \in A, a \leq x$.
- An **lower bound** of A is an element $x \in X$ such that $\forall a \in A, a \geq x$.
- The **supremum** of A is the least upper bound of A.
- The **infimum** of A is the greatest lower bound of A.

Definition 1.3.5. An **interval** on an ordered set $(X, <)$ is

- $(a, b) = \{x \in X : a < x < b\}$ for some $a, b \in X$
- $[a, b) = \{x \in X : a \leq x < b\}$ for some $a, b \in X$
- $(a, b] = \{x \in X : a < x \leq b\}$ for some $a, b \in X$
- $[a, b] = \{x \in X : a \leq x \leq b\}$ for some $a, b \in X$

1.4 Cardinality

Definition 1.4.1. A set A is **finite** if there exists a bijection $f : A \hookrightarrow\!\!\!\!\rightarrow \{1, 2, 3, ..., n\}$ for some $n \in \mathbb{N}$.

Definition 1.4.2. The **cardinality** of a finite set A is $n \in \mathbb{N}$ such that $f : A \hookrightarrow\!\!\!\!\rightarrow \{1, 2, 3, ..., n\}$ is a bijection.

Theorem 1.4.1. Let A be a finite set with cardinality $n \in \mathbb{N}$ and $B \subsetneq A$ be a proper nonempty subset, then

$$\nexists \text{ a bijection } B \hookrightarrow\!\!\!\!\rightarrow \{1, ..., n\}$$

$$\exists \text{ a bijection } B \hookrightarrow\!\!\!\!\rightarrow \{1, ..., m\} \text{ for some } m \in \mathbb{N}$$

Corollary 1.4.1.1. For finite sets A there is no bijection between A and any proper nonempty subset $B \subsetneq A$.

Definition 1.4.3. A set A is **countable** iff $\exists A \hookrightarrow\!\!\!\!\rightarrow \mathbb{N}$ or A is finite.

Theorem 1.4.2. Let A be a nonempty set, then the following are equivalent.

- A is countable
- There exists a surjection $g : \mathbb{N} \twoheadrightarrow A$.
- There exists an injection $f : A \hookrightarrow \mathbb{N}$.

Corollary 1.4.2.1. Every subset $A \subset \mathbb{N}$ is countable.

Corollary 1.4.2.2. A countable union of countable sets is countable.

Definition 1.4.4. The **power set** of a set A denoted $P(A)$ is the set of all subsets of A.

Theorem 1.4.3. The Cantor Theorem states that for a nonempty set A there is no injection $f : P(A) \hookrightarrow A$ and no surjection $g : A \twoheadrightarrow P(A)$.

1.5 Topologies

Definition 1.5.1. A **topology** on a set A is a set of subsets $J \subset P(A)$ with the following properties

1. $\emptyset, A \in J$.
2. Any union of elements in J is also in J.
3. Any finite intersection of elements in J is also in J.

Definition 1.5.2. A **topological space** is a pair $(X, \mathcal{T})$ sometimes denoted $\mathcal{T}_X$ of a set X and a topology $\mathcal{T}$ on X.

Definition 1.5.3. A subset $A \subset X$ is **open** iff $A \in \mathcal{T}$ where $(X, \mathcal{T})$ is a topological space.

Definition 1.5.4. A subset $A \subset X$ is **closed** iff $X - A$ is open.

Definition 1.5.5. A **basis** is a collection $\mathcal{B}$ of subsets of a set X such that

1. $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$.
2. $\forall B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_1$.

Proposition 1.5.1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C} \subset P(X)$. If $\forall U \in \mathcal{T}$, $\forall x \in U$, $\exists D \in \mathcal{C}$ such that $x \in D \subseteq U$, then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Definition 1.5.6. The **topology generated by a basis** $\mathcal{B}$ on a set X is

$$\mathcal{T} = \{U \in P(X) : U = \bigcap_{B_i \in \mathcal{C}} B_i, \mathcal{C} \subset \mathcal{B}\}$$

Definition 1.5.7. A **subbasis** for a topology $\mathcal{T}$ on X is a collection $\mathcal{S}$ of subsets of X such that the collection of all unions and finite intersections of elements in $\mathcal{S}$ is $\mathcal{T}$.

Definition 1.5.8. The **topology generated by a subbasis** $\mathcal{S}$ on a set X is the collection of all unions and finite intersections of elements in $\mathcal{S}$.

Definition 1.5.9. A topology $\mathcal{T}'$ is **finer** than another topology $\mathcal{T}$ iff $\mathcal{T}' \subseteq \mathcal{T}$.

Theorem 1.5.1. Let $\mathcal{B}, \mathcal{B}' \subset P(X)$ be bases of the topological spaces $(X, \mathcal{T}), (X, \mathcal{T}')$. The following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. $\forall x \in X$ and any basis element $B \in \mathcal{B}$ such that $x \in B$ there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

Definition 1.5.10. A **homeomorphism** is a bijection $f : \mathcal{X} \hookrightarrow\!\!\!\!\rightarrow \mathcal{Y}$ between topologies $\mathcal{X}$ and $\mathcal{Y}$.

Definition 1.5.11. A topological space $(X, \mathcal{T})$ is **first countable** iff $\forall x \in X$, $\exists$ a countable set $\mathcal{B} \subset \mathcal{T}$ so that for every set $U \in \mathcal{T}$ containing x, $V \subseteq U$ for some $V \in \mathcal{B}$.

Definition 1.5.12. A topology is **second countable** iff it has a countable basis.

1.5.1 Examples of Topologies

Definition 1.5.13. The **discrete topology** on a set X is $\mathcal{T} = P(X)$.

Definition 1.5.14. The **indiscrete topology** on a set X is $\mathcal{T} = \{\emptyset, X\}$.

Definition 1.5.15. The **finite compliment topology** on a set X is $\mathcal{T} = \{U \subset X : X - U \text{ is finite}\}$.

Definition 1.5.16. The **standard topology** on $\mathbb{R}$ is the topology generated by the basis

$$\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.17. The **lower limit topology** on $\mathbb{R}$ denoted $\mathbb{R}_\ell$ is the topology generated by the basis

$$\mathcal{B} = \{[a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.18. The **upper limit topology** on $\mathbb{R}$ is the topology generated by the basis

$$\mathcal{B} = \{(a, b] \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.19. The **K-topology** on $\mathbb{R}$ denoted $\mathbb{R}_K$ is the topology generated by the basis

$$\mathcal{B} = \{(a,b) \subset \mathbb{R} : a,b \in \mathbb{R}\} \cap \{(a,b) - K \subset \mathbb{R} : a,b \in \mathbb{R}\}$$

$$K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

Definition 1.5.20. The **order topology** on a ordered set S with more than 1 element is the topology generated by the a basis

$$\mathcal{B} = \{(a,b) \subset \mathbb{R} : a,b \in \mathbb{R}\} \cap \{(a,b_0] \subset \mathbb{R} : a \in \mathbb{R}\} \cap \{[a_0,b) \subset \mathbb{R} : b \in \mathbb{R}\}$$

where a_0 is the smallest element and b_0 is the largest element.

1.6 Well Ordered Sets

Definition 1.6.1. A **well ordered set** X is an ordered set such that any subset $S \subseteq X$ has a smallest element $s_0 \in S$ such that $s_0 \leq s$, $\forall s \in S$.

Corollary 1.6.0.1. Any finite ordered set is well ordered.

Definition 1.6.2. The **section** of a well ordered set X by $a \in X$ denoted S_a is

$$S_a = \{x \in X : x < a\}$$

Theorem 1.6.1. Any set A admits a well ordering.

Corollary 1.6.1.1. There exists an uncountable well ordered set.

Theorem 1.6.2. There exists a well ordered set S such that any section is countable S_Ω where Ω is the largest element.

Definition 1.6.3. The **minimal uncountable well ordered set** denoted S_Ω is the uncountable well-ordered set such that any section is countable.

Theorem 1.6.3. If $A \subset S_\Omega$ is a countable subset of S_Ω then A has an upper bound in S_Ω.

1.7 Product Topology

Definition 1.7.1. The **Product Topology** denoted $\mathcal{T} \times \mathcal{T}'$ for two topologies $\mathcal{T}, \mathcal{T}'$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}, V \in \mathcal{T}'\}$$

Theorem 1.7.1. For topologies $\mathcal{T}$ and $\mathcal{T}'$ with bases $\mathcal{B}$ and $\mathcal{B}'$ the product topology is equivalently generated by the basis

$$\mathscr{B} = \{B \times B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$$

Definition 1.7.2. The function $\pi_n : \prod_{i \in I} X_i \to X_n$ is the function mapping $(\ldots, x_n, \ldots) \mapsto x_n$.

Theorem 1.7.2. The product topology on $X \times Y$ is the weakest topology such that $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are both open $\forall$ open $U \subset X$, $V \subset Y$.

Lemma 1.7.3. Let X, Y be topological spaces the set $\{U \times Y : U \in X\} \cap \{X \times V : V \subset Y\}$ is a subbasis for $X \times Y$.

1.8 Subspace Topology

Definition 1.8.1. The **subspace topology** denoted $\mathcal{T}_Y$ for a subset $Y \subset X$ of a topological space $(X, \mathcal{T})$ is

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

Lemma 1.8.1. If $\mathcal{B}$ is a basis for a topological space X and $Y \subset X$ then $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

Lemma 1.8.2. For $Y \subset X$ with the subspace topology, if $U \subset Y$ is open in Y and $Y \subset X$ is open in X then U is open in X.

Theorem 1.8.3. Let $A \subset X$, $B \subset Y$ be topological spaces with subspace topologies, then for $A \times B \subset X \times Y$ the product topology agrees with the subspace topology.

Definition 1.8.2. A subset $Y \subset X$ is **convex** iff $\forall a, b, c \in Y$, if $a < c < b$ then $c \in Y$.

Theorem 1.8.4. If X be an ordered set with a convex subset $Y \subset X$, then the subspace topology on Y is the order topology on Y.

1.9 Interior and Closure

Definition 1.9.1. For a subset $V \subset X$ of a topological space X the **interior** of V denoted V^o is the largest open set in V or equivalently

$$V^o = \bigcup_i U_i \quad \forall U_i \in V$$

Definition 1.9.2. For a subset $V \subset X$ of a topological space X the **closure** of V denoted $\bar{V}$ is the smallest closed set containing V or equivalently

$$\bar{V} = X - (X - V)^o = \bigcap_j F_j \quad \forall F_j \in \mathcal{T} \text{ such that } V \subset F_j$$

Lemma 1.9.1. Let $A \subset X$ be a subset of a topological space X, then $x \in \bar{A}$ if and only if every open set containing $x \in X$ intersects A.

Lemma 1.9.2. Let $A \subset X$ be a subset of a topological space X and $\mathcal{B}$ be a basis for X, then $x \in \bar{A}$ if and only if every basis element $B \in \mathcal{B}$ containing $x \in X$ intersects A.

Definition 1.9.3. For a subset $A \subset X$ of a space X, a point $x \in X$ is a **limit point** or **cluster point** of A iff for every neighborhood $U \in \mathcal{T}$ of x, $U - \{x\}$ intersects A.

Theorem 1.9.3. Let $A \subset X$ be a subset of a topological space X and A' be the set of limit points of A, then $\bar{A} = A \cup A'$.

1.10 Hausdorff Topologies

Definition 1.10.1. A topological space X is **Hausdorff** iff $\forall x, y \in X$ such that $x \neq y$, there exists $U \in \mathcal{T}$ and $V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 1.10.1. Every finite subset of a Hausdorff space is closed.

Definition 1.10.2. A **sequence** in a space X is a series of points $x_i \in X$ for $i \in \mathbb{N}$.

Definition 1.10.3. A sequence **converges** to a point $x \in X$ iff for all open subsets $U \subset X$ such that $x \in U$ $\exists N$ such that for all $n \geq N$, $x_n \in U$.

Proposition 1.10.1. Let $A \subset X$ be a subset of a topological space X. If a sequence $x_n \in A$ converges to $x \in A$, then $x \in \overline{A}$.

Theorem 1.10.2. If a space X is Hausdorff, then any sequence $x_n \in X$ can only converge to at most one point.

1.11 Continuity

Definition 1.11.1. A function $f : X \to Y$ is **continuous** iff for any open subset $V \subset Y$ in the range of f, $f^{-1}(V)$ is open.

Proposition 1.11.1. A function $f : X \to Y$ is continuous iff for $\mathcal{B}$ basis of Y, $f^{-1}(\mathcal{B})$ is open $\forall B \in \mathcal{B}$.

Theorem 1.11.1. If $f : \mathbb{R} \to \mathbb{R}$ is continuous under the epsilon delta definition from real analysis, then it is continuous.

Definition 1.11.2. A **homeomerphism** is a function between spaces that is continuous in both directions.

Theorem 1.11.2. Let $f : X \to Y$ a function between spaces. The following are equivalent"

- f is continuous.
- $\forall A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
- $\forall B \subset Y$ closed, $f^{-1}(B)$ is closed
- $\forall x \in X$, if $f(x) \in B$ is a neighborhood of $f(x)$ then $f^{-1}(B)$ contains a neighborhood of x.

Theorem 1.11.3. Let X, Y, Z be spaces.

- $f : X \to Y$ defined by $x \mapsto y$ for some $y \in Y$ and $\forall x \in X$, is continuous.
- For $A \subset X$ with the subspace topology, $j : A \to X$ defined by $x \mapsto x$ is continuous.
- If $f : X \to Y$ and $g : Y \to Z$ are both continuous then $g \circ f = g(f(x))$ is continuous.
- If $f : X \to Y$ is continuous. For $A \subset X$, the restriction $f_A : A \to Y$ is continuous.
- For $Y \subset Z$ with the subspace topology, if $f : X \to Y$ is continuous then $f : X \to Z$ is also continuous.
- The map $f : X \to Y$, where $X = \bigcup_\alpha U_\alpha$ for open subsets U_α is continuous if and only if $f_{\bigcup_{\alpha \in A} U_\alpha}$ is open for all A.

Theorem 1.11.4. The Pasting Theorem states that for $X = A \cup B$ where $A, B \subset X$ are closed subsets. If $f : A \to Y$ is continuous, $g : B \to Y$ is continuous and $f = g$ on $A \cap B$, then $h : X \to Y$ is continuous where

$$h(x) = \left\{ \begin{array}{ll} f(x) & x \in A \\ g(x) & x \in B \end{array} \right\}$$

Theorem 1.11.5. Let $f : X \to Y \times Z$ for spaces X, Y, Z where $f(x) = (f_1(x), f_2(x))$. f is continuous iff f_1, f_2 are continuous.

Proposition 1.11.2. The functions $+, \times, /$ are continuous, defined by

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{defined by } (a, b) \mapsto a + b$$
$$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{defined by } (a, b) \mapsto ab$$
$$/ : \mathbb{R} \times \mathbb{R} - \{0\} \to \mathbb{R} \quad \text{defined by } (a, b) \mapsto a/b$$

1.12 Metric Spaces

Definition 1.12.1. A **metric space** X is a set with a function $d : X \to \mathbb{R}$ such that $\forall x, y, z \in X$,

- $d(x, y) \geq 0$
- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) + d(y, z) \leq d(x, z)$

Definition 1.12.2. The **metric ball** denoted $B(x, \varepsilon)$ for a point $x \in X$ in a metric space (X, d) and a real number $\varepsilon > 0$ is the set

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

Definition 1.12.3. The **topology on a metric space** is the topology generated by the basis

$$\mathcal{B} = \{B(x, \varepsilon) : x \in X, \varepsilon > 0)\}$$

Definition 1.12.4. A topological space X is **metrizable** iff $\exists$ a metric on X whose metric topology is that of X.

Theorem 1.12.1. For two metrics (X, d) and (X, d'), the metric topology generated by d' is finer than d if and only if $d(x, y) \leq d'(x, y)$

Corollary 1.12.1.1. For two metrics (X, d) and (X, d') with metric topologies $\mathcal{T}$ and $\mathcal{T}'$. $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $x \in X$ and all $\varepsilon > 0 \in \mathbb{R}$, $\exists\ \delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$

Definition 1.12.5. A subset set $A \subseteq X$ of a metric space X is bounded iff $\exists k > 0 \in \mathbb{R}$ so that $d(x, y) < k$ for all $x, y \in A$.

Definition 1.12.6. The diameter of a metric space is $\sup_{x,y \in A} d(x, y)$.

Theorem 1.12.2. Let (X, d) be a metric space, then $\overline{d} : X \times X \to \mathbb{R}$ defined by $\overline{d}(x, y) = \min\{d(x, y), 1\}$ is a metric that induces the same topology.

Proposition 1.12.1. Let $A \subset X$, where X is a metric $\forall x \in \overline{A}$, $\exists$ a sequence $x_n \in A$ for $n \in \mathbb{N}$ that converges to X.

Theorem 1.12.3. Let X be a metric space and $f : X \to Y$ a function. f is continuous if and only if $\lim f(x_n) = f(x)$ for all sequences $x_n \in X$ that converge to $x \in X$.

Proposition 1.12.2. Every metrizable space is first countable.

Theorem 1.12.4. Let X, Y be spaces and $f : X \to Y$ a function.
If f is continuous, then for every sequences $x_n \in X$ converging to $x \in X$, $f(x_n)$ converges to $f(x)$.
If X is first countable and for every sequence $x_n \in X$ converging to $x \in X$, $f(x_n)$ converges to $f(x)$, then f is continuous.

Chapter 2

Connectedness and Compactness

2.1 Product Topology

Definition 2.1.1. Let I be an index set, an **I-tuple** in a set X is a function $x : I \to X$ denoted $x_i = f(i)$ for $i \in I$.

Definition 2.1.2. The **Cartesian product** $\prod_{i\in I} A_i$ for a family of sets $\{A_i\}_{i\in I}$ is the set of $x : J \to \cup_{i\in I} A_i$ where $x(i) \in A_i$.

Definition 2.1.3. The **box topology** on $\prod_{i\in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{i\in I} U_i : U_i \in \mathcal{T}_{X_i} \right\}$$

Definition 2.1.4. The **product topology** $S_{\mathcal{B}} \subset P(\prod_{i\in I} X_i)$ is the topology generated by the subbasis

$$\mathcal{B} = \left\{ \prod_{\substack{i\in I \\ i\neq j}} X_i \times U_j : \text{open } U_j \subset X_j \right\}$$

Corollary 2.1.0.1. π_i is continuous for a topology $\mathcal{T}$ on $\prod_{i\in I} X_i$ iff $\mathcal{T}$ contains the product topology.

Theorem 2.1.1. The product topology is the smallest topology for which π_i is continuous for all $i \in I$.

Theorem 2.1.2. $\mathbb{R}^\omega$ with the product topology is metrizable with the metric $d(\mathbf{x}, \mathbf{y}) = \overline{d}(x_1, y_1) + \frac{1}{2}\overline{d}(x_2, y_2) + \cdots + \frac{1}{n}\overline{d}(x_n, y_n)$.

2.2 Connectedness

Definition 2.2.1. A **separation** of a space X is a pair of disjoint open subsets $U, V \subset X$ such that $X = U \cup V$.

Definition 2.2.2. A space X is **connected** iff it has no separation.

Definition 2.2.3. A space X is **disconnected** iff it has a separation.

Definition 2.2.4. A space is **path connected** iff for any $x, y \in X$, there exists a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$.

Theorem 2.2.1. $\mathbb{R}$ is connected and the intervals of $\mathbb{R}$ are connected.

Proposition 2.2.1. Let $X \cong Y$. X is connected $\Leftrightarrow$ Y is connected.

Theorem 2.2.2. If $f : X \to Y$ is continuous and X is connected then $f(X)$ is connected.

Definition 2.2.5. A topological space X is **homogeneous** iff there exists a homeomorphism $f : X \to X$ such that $f(x) = y, \forall x, y \in X$.

Corollary 2.2.2.1. Let X be a space and $Y \subset X$ be a connected subspace. If $A \cup B = X$ is a separation of X then either $Y \subset A$ or $Y \subset B$.

Theorem 2.2.3. Let $\{A_\alpha\}$ be a collection of connected sets in a space X with $\bigcap_\alpha A_\alpha \neq 0$ then $\bigcup_\alpha A_\alpha$ is connected.

Theorem 2.2.4. Let $A \subset X$ be a connected space with the subspace topology on a space X. If $A \subseteq B \subseteq X$ then A is connected with the subsapce topology as a subset of B.

Theorem 2.2.5. The product $X \times Y$ with the product topology of two connected spaces X and Y is connected.

Theorem 2.2.6. The Intermediate Value Theorem states that for a continuous function $f : X \to Y$ were X is connected and y is ordered, if $a, b \in X$, $\forall y\ inY$ such that $f(a) \leq y \leq f(b)$, there exists $c \in X$ such that $f(c) = y$.

2.3 Linear Continua

Definition 2.3.1. A **linear continua** is an ordered space L such that the following properties hold

1. L has the least upper bound property
2. $\forall x, y \in L$ such that $x < y$, $\exists Z$, such that $x < z < y$.

Theorem 2.3.1. A linear continua and intervals of linear continua are connected.

2.4 Components

Definition 2.4.1. The **component** of x in a space X is the union of all connected subsets $S \subseteq X$ containing x.

Corollary 2.4.0.1. The components of a space are connected and disjoint.

Definition 2.4.2. The **path component** of x in a space X is the set of points $y \in X$ such that there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Corollary 2.4.0.2. The path components of a space are connected and disjoint.

2.5 Local Connectedness

Definition 2.5.1. A space X is **locally connected** at $x \in X$ iff for all open neighborhoods $U \subset X$ of $x \in U$, there exists a connected open subset $V \subseteq U$.

Theorem 2.5.1. A space is locally connected iff the components of elements in open sets are open.

Corollary 2.5.1.1. A connected space is locally connected.

Definition 2.5.2. A space X is **locally path connected** at $x \in X$ iff for all open neighborhoods $U \subset X$ of $x \in U$, there exists a path connected open subset $V \subseteq U$.

Theorem 2.5.2. A space is **locally path connected** iff the path components of elements in open sets are open.

Corollary 2.5.2.1. A path connected space is locally path connected.

Theorem 2.5.3. If a space is locally path connected, then path components and components are equivalent.

2.6 Compactness

Definition 2.6.1. A **cover** of a set X is collection of subsets such that the union contains X.

Definition 2.6.2. A **sub-cover** of a set X is a subset of a cover that is also a cover.

Definition 2.6.3. A space X is **compact** iff every open cover has a finite sub-cover.

Definition 2.6.4. A space X is **sequentially compact** iff every sequence has at least one limit point.

Theorem 2.6.1. Closed subsets of compact sets are compact.

Theorem 2.6.2. Compact subset of Hausdorff spaces are closed.

Corollary 2.6.2.1. If a compact Hausdorff space, a subset is compact if and only if it is closed.

Theorem 2.6.3. The image of a compact space under a continuous map is compact.

Corollary 2.6.3.1. Compactness is preserved under homeomorphism.

Theorem 2.6.4. For a continuous bijection $f : X \hookrightarrow\!\!\!\!\rightarrow Y$, if X is compact and Y is Hausdorff then f is a homeomorphism.

Theorem 2.6.5. The product of two compact spaces is compact.

2.7 Limit Point Compactness

Definition 2.7.1. A space X is **limit point compact** if every infinite subset has a limit point.

Theorem 2.7.1. Compact spaces are limit point compact.

Corollary 2.7.1.1. Closed intervals of $\mathbb{R}^n$ are limit point compact.

Theorem 2.7.2. Let X be a metrizable space, then compactness and limit point compactness are equivalent.

2.8 Local Compactness

Definition 2.8.1. A space X is **locally compact** at $x \in X$ iff there exists a compact $K \subset X$ with $x \in K$ such that K contains a neighborhood of x.

Corollary 2.8.0.1. A compact space is locally compact.

Corollary 2.8.0.2. $\mathbb{R}^n$ is locally compact.

Definition 2.8.2. A **1-point compactification** of a space X is a superset $Y \supset X$ such that

- $Y - X$ is a single element.
- Y is compact.

Theorem 2.8.1. A space X is locally compact and Hausdorff if and only if it has a Hausdorff 1-point compactification.

Theorem 2.8.2. If a 1-point compactification exists it is unique.

Theorem 2.8.3. Let X be a Hausdorff space then X is locally compact if and only if for all open neighborhoods $U \subset X$ of a point $x \in X$, there exists an open neighborhood $V \subset X$ of x such that $V \subset U$ and $\overline{V}$ is compact.

Chapter 3

The Fundamental Group

3.1 Quotient Maps

Definition 3.1.1. A **quotient map** $f : X \to Y$ between two spaces X and Y is a continuous map such that $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

Definition 3.1.2. The **quotient topology** of a surjective map $f : X \to Y$ is the unique topology of Y such that f is a quotient.

Proposition 3.1.1. $f : X \to Y$ is a quotient if and only if $U \subset Y$ is closed if and only if $f^{-1}(U) \subset X$ is closed.

Theorem 3.1.1. A function $f : X \to Y$ is continuous if and only if the function from $\overline{f} : X/\sim \to Y$ defined by $[x] \mapsto f(x)$ and $[x] = \{z \in X : f(z) = f(x)\} \subset X/\sim$ is continuous.

3.2 Homotopy

Definition 3.2.1. Two functions $f_1, f_2 : X \to Y$ are **homotopic** iff there exists a function $f : X \times I \to Y$ such that $f(x, a) = f_1(x)$ and $f(x, b) = f_2(x)$.

Definition 3.2.2. Two functions $f_1, f_2 : I \to X$ with $f_1(a) = f_2(a) = x_0$, $f_1(b) = f_2(b) = x_1$ are **path homotopic** iff there exists $f : I \times It\ X$ such that $f(x, a) = f_1(x)$, $f(x, b) = f_2(x)$, $f(a, t) = x_0$ and $f(b, t) = x_1$ for all $t \in I$.

Corollary 3.2.0.1. If two functions are path homotopic then they are homotopic.

Proposition 3.2.1. Homotopy and path homotopy are equivalence relations.

Definition 3.2.3. The **homotopy classes** of paths in X from $x_0 \in X$ to $x_1 \in X$ denoted $\pi(x_0, X, x_1)$ is the equivalency classes of all path homotopic functions from x_0 to x_1.

Theorem 3.2.1. Let $A \subset \mathbb{R}^n$ be a convex subset. Two functions $f_1, f_2 : I \to A$ have the same endpoints if and only if f_1 and f_2 are path homotopic.

3.3 Path Composition

Definition 3.3.1. The **path composition** $f_1 * f_2 \in \pi(x, X, z)$ of two paths $f_1 \subset \pi(x, X, y)$ and $f_2 \subset \pi(y, X, z)$ is the path defined by pasting two paths together at $f_1(1) = f(0)$.

Theorem 3.3.1. There exists a well defined mapping $f : \pi(x_1, X, x_2) \times \pi(x_2, X, x_3) \to \pi(x_1, X, x_3)$.

Definition 3.3.2. The **identity path** denoted e_x for a point $x \in X$ is the path $e_x : I \to X$ such that $e_x(t) = x$ for all $t \in I$.

Definition 3.3.3. The **inverse path** denoted $\overline{f} \in \pi(y, X, x)$ of a path $f \in \pi(x, X, y)$ is the path $\overline{f} = (f * z)$ where $z : I \to I$ is defined by $t \mapsto b - t$.

Theorem 3.3.2. Path composition of homotopy classes have the following properties for any three path homotopy classes $\gamma_1 = \pi(x_1, X, x_2)$, $\gamma_2 = \pi(x_2, X, x_3), \gamma_3 = \pi(x_3, X, x_4)$

1. $\gamma_3 * (\gamma_2 * \gamma_1) = (\gamma_3 * \gamma_2) * \gamma_1$.
2. $\overline{\gamma_1} * \gamma_1 \cong e_{x_1}$ and $\gamma_1 * \overline{\gamma_1} \cong e_{x_2}$.
3. $\gamma_1 * e_{x_1} = \gamma_1$ and $e_{x_2} * \gamma_1 = \gamma_1$.

3.4 The Fundamental Group

Definition 3.4.1. The **fundamental group** denoted $\pi_1(X, x_0)$ for a space X and a point $x_0 \in X$ is the group of path homotopy classes of loops based at x_0 equipped with path composition.

Proposition 3.4.1. The fundamental group is a group.

Theorem 3.4.1. Let $f : X \to Y$ be a homeomorphism then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

Corollary 3.4.1.1. The fundamental group is isomorphic under homeomorphism.

Definition 3.4.2. The **base translation isomorphism** $\hat{f}$ for a path $f : I \to X$ in X from $f(0) = x_0 \in X$ to $f(1) = x_1 \in X$ is a map $\hat{f} : \pi_1(X, x_0) \to \pi(X, x_1)$ defined by

$$\hat{f}([g]) = [\overline{f}] * [g] * [f]$$

Theorem 3.4.2. The base translation isomorphism is an isomorphism.

Corollary 3.4.2.1. If a space X is path-connected, then the fundemental group does not depend on choice of base point, $\pi_1(X, x_0) \cong \pi_1(X, x_1), \forall x_0, x_1 \in X$.

3.5 Covering Maps

Definition 3.5.1. An open set $U \subseteq X$ is **evenly covered** by a continuous subjective map $p : E \to X$ iff $p^{-1}(U)$ is a disjoint union of open sets $p^{-1}(U) = \bigcup_\alpha V_\alpha$ so that p restricted to each V_α is a homeomorphism.

Definition 3.5.2. A **covering map** is continuous subjective map $p : E \to X$ such that a neighborhood around every point is evenly covered.

Theorem 3.5.1. Let $p : E \to X$ be covering map and $X_0 \subset X$ a subspace with the subspace topology, then $p|_{p^{-1}(X_0)}$ a covering map of X_0.

Theorem 3.5.2. Let $p : E \to X$ and $p' : E' \to X'$ be covering maps, then $p \times p' : E \times E' \to X \times X'$ is a covering map.

3.6 Liftings

Definition 3.6.1. The **lifting** $\tilde{f}$ of two continuous maps $f : I \to X$ and $p : E \to X$ is a continuous map $\tilde{f} : I \to E$ such that $p\tilde{f} = f$.

Theorem 3.6.1. Let $f : I \to X$ be a continuous map and $p : E \to X$ be a covering map with $e_0 \in p^{-1}(x_0)$ for $x_0 = f(0)$, then there exists a unique lifting $\tilde{f} : I \to E$ so that $\tilde{f}(0) = e_0$.

Proposition 3.6.1. Let $f : I \times I \to X$ be a continuous map and $p : E \to X$ be a covering map with $e_0 \in p^{-1}(x_0)$ for $x_0 = f(0)$, then there exists a unique lifting $\tilde{f} : I \times I \to E$ so that $\tilde{f}(0, 0) = e_0$.

Theorem 3.6.2. Let $p : E \to X$ be a covering map where $p(e_0) = x_0 \in X$. For any two homotopic paths $f, g : I \to X$, the liftings $\tilde{f}$ and $\tilde{g}$ with endpoint e_0 are homotopic.

Definition 3.6.2. The **lifting correspondence** Φ for a covering map $p : E \to X$ is a mapping $\Phi : \pi_1(X, x_0) \to p^{-1}(b)$ is defined for $[f] \in \pi_1(X, x_0)$ with a unique homotopy class of liftings $\tilde{f} : I \to E$ such that $\Phi(f) = \tilde{f}(1)$.

Definition 3.6.3. A topological space is **simply connected** if it has the fundamental group $\{1\}$ for all base points.

Corollary 3.6.2.1. A convex space is simply connected.

Theorem 3.6.3. Let $p : E \to B$ be a covering map, and let $p(e_0) = x_0$. If E is path connected then the lifting correspondence is surjective. If E is simply connected, then it is bijective.

Theorem 3.6.4. For a covering map $p : E \to B$ and base point $p(e_0) = x_0 \in X$.

1. The induced map $p_* \pi_1(E, e_0) \to \pi_1(X, x_0)$ is injective.
2. The lifting correspondence induces an injective map $\theta : \pi_1(X, x_0)/\pi_1(E, e_0) \top^{-1}(x_0)$, which is bijective if E is path connected.
3. If f is a loop in X based at x_0, then $[f] \in \pi_1(E, e_0)$ if and only if f lifts to a loop in E based at e_0.

3.7 Retractions

Definition 3.7.1. A **retraction** of X onto a subset $A \subseteq X$ is a continuous map $r : X \to A$ so that $r|_A$ is the identity.

Definition 3.7.2. A subset $A \subseteq X$ is a **retract** of X iff there exists a retraction of X onto A.

Proposition 3.7.1. If A is a retract of X, then the induced homomorphism of fundamental groups is injective.

Theorem 3.7.1. There is no retraction of B^2 onto S^1.

Definition 3.7.3. A path $h : I \to X$ is **null homotopic** iff it is homotopic to a constant map.

Theorem 3.7.2. Let $j : S^1 \to X$ be a continuous map, then the following are equivalent:

- h is null homotopic.
- h extends to a map $k : B^2 \to X$.
- The induced homomorphism of fundamental groups is the zero map.

Theorem 3.7.3. The Bauer fixed point theorem states that if $f : B^n \to B^n$ is a continuous map between open balls, then there exists a point $x \in B^n$ such that $f(x) = x$.

"At least if you believe in calculus."

"I hope that homework didn't kill anyone too much."

"It's like set theory, but actually interesting!"

"If you want to make class more interesting just replace 'bases' with Al-Qaeda."

"If you're trapped in the U-ball, you're screwed!"

"It's in all these weird languages that nobody should be speaking with too many consonants."

"Hmm, maybe"

"The inverse function theorem is basically the reason the world works."

"That has nothing to do with this... although it may."

"Whenever an arbitrary collection is involved, you are at risk of set theory."

"I'm going on strike because things cost too much in France."

"No-one remembers the green zone."

"Once you've obtained mathematical adulthood, you have to know this example."

"I'm going to regret erasing this, but then I get to have fun drawing it again!"

"This proof is not moral."

"If your theorems are wrong change your definitions so your theorems are right!"

"Then he became a philosopher, which is very sad."